

Problem

Let ABC be a triangle and E the orthocenter. The bissectrices from B, C of the angles $\widehat{ABE}, \widehat{ACE}$ are intersected to the point F . AD is the diameter of the circumcircle (GR) of ABC . Let $FPDQ$ the orthogonal quadrilateral with sides FQ, PD parallel to the bissectrice of the angle A . Prove that the diagonal PQ of $FPDQ$ bissects BC .

see the figure

Solution:

1. E the orthocenter of ABC (common point of altitudes)
2. F the point of intersection of the bissectrice of $\widehat{ABE}, \widehat{ACE}$.
3. LA' the orthogonal diameter to BC
4. $RL \perp AA', QD \perp AA', FP \perp LD, BI \perp AA', BI' \perp LD$
5. $\widehat{AAD} = \vartheta = \frac{B-C}{2}, \widehat{EAO} = B-C, MM' \perp LD$
6. $\widehat{BFC} = A + \widehat{ABF} + \widehat{ACF} = A + \frac{1}{2}(90^\circ - A) + \frac{1}{2}(90^\circ - A) = 90^\circ$

(A)

$$LM = 2R - \frac{a}{2} \operatorname{tg} \frac{A}{2} = 2R - \frac{1}{2} 2R \sin A \cdot \operatorname{tg} \frac{A}{2} = 2R \left(1 - \sin^2 \frac{A}{2}\right) = 2R \cos^2 \frac{A}{2}$$

$$LM' = LM \cos \vartheta = LM \cos \frac{B-C}{2}$$

$$PM' = \frac{a}{2} \sin \left(90 - \frac{B-C}{2}\right) \text{ because } \widehat{MFP} = \widehat{MCF} = 90 - \frac{B-C}{2}$$

$$MN' = LM \sin \vartheta = LM \sin \frac{B-C}{2} = 2R \cos^2 \frac{A}{2} \sin \frac{B-C}{2}$$

$$\frac{MM'}{PM'} = \frac{2R \cos^2 \frac{A}{2} \sin \frac{B-C}{2}}{\frac{a}{2} \cos \frac{B-C}{2}} = \operatorname{ctg} \frac{A}{2} \operatorname{tg} \frac{B-C}{2}$$

$$\Rightarrow \operatorname{tg} \widehat{MPD} = \operatorname{ctg} \frac{A}{2} \operatorname{tg} \frac{B-C}{2} \quad (1)$$

(B)

$$PD = FQ = FM'' + M''Q$$

$$FM'' = \frac{a}{2} \cos \frac{B-C}{2} \quad \text{from } FM''M$$

$$QM'' = \frac{a}{2} \operatorname{tg} \frac{A}{2} \cdot \cos \frac{B-C}{2} \quad \text{from } FM''M$$

$$\text{That is: } FQ = \frac{a}{2} \cos \frac{B-C}{2} + \frac{a}{2} \operatorname{tg} \frac{A}{2} \cos \left(\frac{B-C}{2} \right) = \frac{a}{2} \cos \frac{B-C}{2} \left[1 + \operatorname{tg} \frac{A}{2} \right]$$

$$M''M' = M''M + MM' = \frac{a}{2} \sin \frac{B-C}{2} + LM \sin \frac{B-C}{2} =$$

$$\frac{a}{2} \sin \frac{B-C}{2} + 2R \cos^2 \frac{A}{2} \cdot \sin \frac{B-C}{2} = 2R \sin \frac{B-C}{2} \cos \frac{A}{2} \left[\sin \frac{A}{2} + \cos \frac{A}{2} \right]$$

$$\frac{FP}{FQ} = \frac{M''M'}{FQ} = \frac{2R \sin \frac{B-C}{2} \cos \frac{A}{2} \left[\sin \frac{A}{2} + \cos \frac{A}{2} \right]}{\frac{a}{2} \cos \frac{B-C}{2} \left[1 + \operatorname{tg} \frac{A}{2} \right]} =$$

$$= \frac{4R \sin \frac{B-C}{2} \cos \frac{A}{2} \left[\sin \frac{A}{2} + \cos \frac{A}{2} \right]}{4R \sin \frac{A}{2} \cos \frac{A}{2} \left[1 + \operatorname{tg} \frac{A}{2} \right]} = \operatorname{tg} \frac{B-C}{2} \operatorname{cotg} \frac{A}{2}$$

$$\text{That is } \operatorname{tg} \hat{QPD} = \operatorname{tg} \frac{B-C}{2} \operatorname{cotg} \frac{A}{2}$$

(2)

$$\text{From (1), (2)} \Rightarrow \hat{MPD} = \hat{QPD}$$

