A Minkowski plane with a Radon curve as unit circle.

G. A. Tsintsifas

The study of the Normed linear spaces some times is easier after some Geometrization of the space. It is well known that every Normed linear space E is characterized by the two-suspace of E, that is by its M^2 Minkowski plane. An interesting class of Minkowski planes are those who accept as unit circle the Radon curves. The study of the Radon curves as unit curves in Minkowski Geometry is very interesting and supplies some Geometrical characterizations for Euclidean spaces. We usually call as inner product space an infinite Normed linear space provided by an inner product property and the finite Normed space Euclidean space.
In this paper, after of an brief introduction to the N^2 plane and to Radon curves we will prove some propositions characteriging the Euclidean spaces.
Let K be a centrosymmetric convex body in E^2. We denote by $V = \vartheta K$ and O the center of K.We suppose $x \in E^2$ and $x' = Ox \cap V$. The Minkowski N^2 plane is defined by the norm

$$\frac{|Ox|}{|Ox'|}$$

Obviously $K = \{x/||x|| \leq 1\}$.
We define the convex set K as unit circle of M^2 and denote K_0 and $V_0 = \vartheta K_0$. The norm axioms are satisfied, as we can see in several books see e.g.[2]. For the simplicity, we suppose that the normality is unique, that is the perpendicular from a point to a line is unique. This is equivalent to the unique tangent (support line) to every point of V_0. We also see that the normality is no symmetric relation, that if is a line is l perpendicular to the line p generally the line p is no perpendicular to the line l. We denote

$$l \dashv p.$$

We denote by x instand $\vec{Ox}$, so we write

$$x \dashv y$$

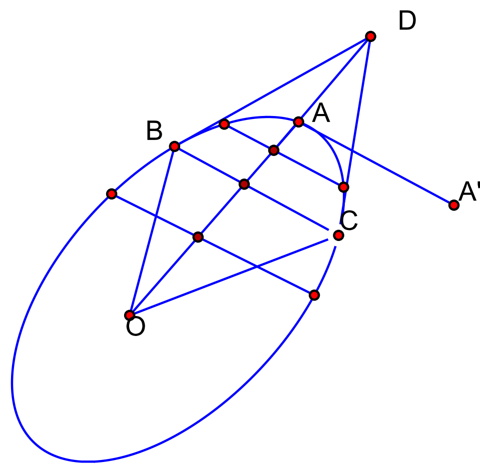
that is $Ox \dashv Oy$.
The symmetrical perpendicularity is defined by:

$$(x \dashv y) \bigwedge (y \dashv x)$$

and we will denote by

$$x \top y$$

The problem of the characterization of a normed linear space as in inner product space is very interesting.
Some well known Geometrical criteria, old and news, are the following.
1. If in the L.normed space E, every two dimensional subspace is Euclidean then E is an inner product space.
2. Suppose that for the n-L.normed space E,for every 3-space $P \in E$, there is a symmetrical perpedicularity, then E is inner product space.
3. If in M^2 every Minkowski circle is ellipse, then M^2 is E^2.
4. Suppose that in M^2 for every $x, y \in M^2$ holds: $x + y \dashv x - y$, then $M^2 = E^2$.
This is an excelent theorem of Thomson. A short proof is the foillowing.
We can easily prove that the unit Minkowski circlel V_0 is smooth. Let A be the tangent point to V_0 of the line q parallel to a given direction $\vec{a}$. BC is a chord of V_0 parallel to $\vec{a}$, see fig 1. We see that BC is parallel to q therefore the diagonal OD of the parallelogramme $OBDC$ is perpendicular to BC and bisects BC. So according the Bertrand theorem see [2] V_0 is an ellipse.



$fig1$

A step closer from the Minkowski Geometry to Euclidean Geometry is the Minkowski Geometry with symmetric perpendicularity. As my impression is in this subject only a few things are known. The unit circle V_0 in M^2 with symmetrical perpendicularity is a Radon curve from the name of the Austrian Mathematicien see [11] who introduced these curves. We shortly explain what is a Radon curve.
We consider an orthogonal Cartesian system xOy and the points $A(1,0)$, $B(0,1)$. Let c be a convex curve in the first quadrant with A and B extrimities. Let $M \in c$ and $r(M) = r(c, \theta)$, where $\vec{r}(M) = \vec{O}M$, that is the $r(M)$ is the length of $\vec{O}M$ in the direction $(cos\theta, sin\theta)$ for $0 \leq \theta \leq \pi/2$.
So we have :

$$c = \{M/r(M) = r(c, \theta)\}$$

We now denote by $h(\theta)$ the support funnction o fc relative to the direction $(cos\theta, sin\theta)$ that is the distance of the orizin O to the support line of c at the point M see fi.2 The polar dual of c is a convex curve denoted by c^* with extrimities A and B, so that

$$h(c^*, \theta).r(c, \theta) = 1$$
$$r(c^*, \theta).h(r, \theta) = 1$$

in the fig. 2 we see that:

$$h(c, \theta) = ON, \qquad r(c^*, \theta) = OM'$$

Therefore:

$$c^* = \{M'/ON.OM' = 1\}$$

The rotation of C^* through O for $\pi/2$ gives a curve $c^*(\pi/2)$ in the second quadrant with extrimities $B(0, 1)$ and $A'(-1, 0)$. In the above points the support lines of $c^*(\pi/2)$ are respectively perpendicular to the axes Oy, Ox. Let now T the image of M' with the above rotation. We easily recognise that the support lines to the points M, T and the lines OM, OT define the parallelogramme $OTEM$. The closed convex curve V_0

$$V_0 = c \bigcup c^*(\pi/2) \bigcup (-c) \bigcup (-c^*(\pi/2))$$

is a Radon curve and the Minkowski space M^2 with unit curve V_0 has the symmetrical perpenticularity

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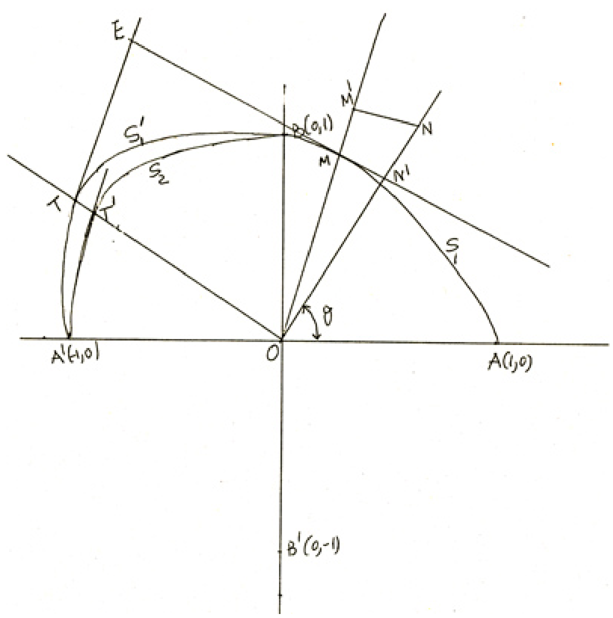
$fig2$



The purpose of our paper is to find criteria of L.normed spaces to be inner product space, but here we use Geometrical methods working with a Radon curve.

Thorem 1

Let $x, y \in M^2$ and V_0 a Radon curve unit circle of M^2. We suppose that

$$x \top y, \qquad \|x\| = \|y\|, \qquad |x|^2 + |y|^2 = constand$$

then the V_0 is circle.

Proof

Let $\vec{x} = \vec{O}M$ and $\vec{y} = \vec{O}T$. we see that:

$$|OM|^2 + |OT|^2 = |OA|^2 + |OB|^2 = 2 \tag{1}$$

The area of the parallelogramme $OTEM$ is equal to

$$(OTEM) = |OT|.|ON| = |OM'|.|ON| = r(c^*, \theta).h(c, \theta = 1$$

Also $(OTEM) = |OM|.|OT||sin\angle MOT|$

that is

$$|OM|.OT||sin\angle MOT| = 1$$

obviously

$$\frac{|om|^2 + |OT|^2}{2} \geq |OM||OT|$$

therefore

$$\frac{|ON|^2 + |OT|^2}{2}|sin\angle MOT| \geq 1 \tag{2}$$

From (1),(2) follows that: $|sin\angle MOT| \geq 1$ That is $\angle MOT = \pi/2$. So the points $M = N$ and we conclude that V_0 is circle and $M^2 = E^2$.

Theorem 2.

Let E be a L.Normed space with symmetrical perpendicularity. We suppose that every square has the diagonals perpedicular and equal. Then E is an inner product space.

Proof

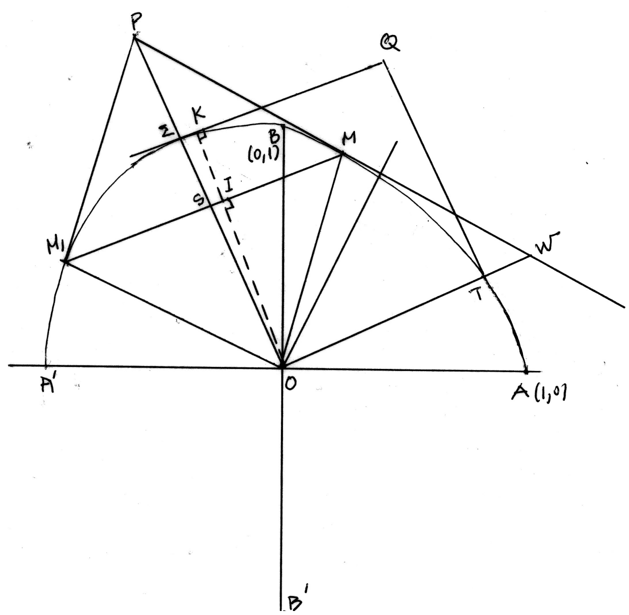
The equivalent problem is: In M^2 the unit circle is the Radon curve V_0. We have to proof if

$$x, y \in V_0 \quad and \quad (x \top y) \bigwedge ((x+y) \top (x-y)) \quad and \quad \|x+y\| = \|x-y\|$$

then $M^2 = E^2$
Let it be :

$$x = M, \quad y = M_1, \quad P = x+y, \quad \Sigma = OP \cap V_0, \quad \|x-y\| = \|OW\| = \|M_1M\|$$

and $T = OW \cap V_0$ see fig 3.



$fig3$

We will have:

$$\|x+y\| = \frac{|OP|}{|O\Sigma|}, \qquad \|x-y\| = \frac{|OW|}{|OT|}$$

But $\|x+y\| = \|x-y\|$ Therefore ΣT is parallel to PW.So we have:

$$\frac{|OS|}{|O\Sigma|} = \frac{\|x+y\|}{2} \tag{3}$$

$$\frac{|SM|}{|\Sigma Q|} = \frac{|SM|}{|OT|} = \frac{\|x-y\|}{2} \tag{4}$$

From (3),(4) follows

$$\frac{|OS|}{|O\Sigma|} = \frac{|SM|}{|\Sigma Q|}$$

hence the point M is in the line OQ. We know from the Radon curve V_0, that $(OM_1PM) = (O\Sigma QT) = 1$ Therefore

$$\frac{|O\Sigma|^2}{|OS|^2} = \frac{(O\Sigma Q)}{(OSM)} = 2$$

, So

$$\frac{|O\Sigma|}{|OS|} = \sqrt{2}$$

That is

$$O\Sigma = \sqrt{2}\frac{x+y}{2}$$

We assume now that the point x is moving in V_0. We can put

$$x = x(\theta), \qquad 0 \le \theta \le 2\pi$$

The curve V_0 supposed to be smooth, so there exists the first derivate Also for simplicity the vector $O\Sigma$ will denoted by u. So we will have:

$$\dot{u} = \frac{\dot{x}+\dot{y}}{\sqrt{2}}$$

But $\dot{u}$ is parallel to the vector M_1M, thus

$$\dot{u} = \frac{\dot{x}+\dot{y}}{\sqrt{2}} = \kappa(x-y)$$

where κ real number.Also for real numbers m, n have

$$\dot{x} = my \qquad \dot{y} = nx \tag{5}$$

that is

$$\frac{my+nx}{\sqrt{2}} = \kappa x - \kappa y,$$

from the above
$$\frac{m}{\sqrt{2}} = -k, \quad \frac{n}{\sqrt{2}} = k$$
that is $m = -n$ and from (5) we take

$$x\dot{x} + y\dot{y} = 0 \quad or \quad x^2 + y^2 = constand$$

So according the theorem 1 we see that V_0 is circle and the L. Normed space is Euclidean.

Corollary 1

In M^2 with symmetrical perpendicularity we assume that V_0 is the unit circle. Suppose that $x, y \in V_0$ and

$$x \top y, \quad ((x+y) \top (x-y)) \bigwedge (\|x-y\|)$$

then it holds
$$\|x+y\|\|x-y\| = 2$$

Proof.

Indeed, let p_1 and p_2 the distances of the center O from M_1M and from the support line at the point S, see fig 3. We will have

$$p_1|x-y| = (OM_1PM) = 1 \tag{6}$$

Also
$$p_2|OT| = (O\Sigma QT) \tag{7}$$
Because of $O\Sigma \quad \top \quad OT$. From (1) and (2) follows that

$$\frac{p_1}{p_2} = \frac{|x-y|}{|OT|} = 1$$

or
$$\frac{|OS|}{O\Sigma|} \cdot \|x-y\| = 1$$

or
$$\frac{2|OS|}{|O\Sigma|} \cdot \|x-y\| = 2$$

and finally $\|x-y\|\|x+y\| = 2$

Corollary 2

If in a L.normed space E with symmetrical perpendicularity the square $ABCD$ with unit sides has perpendicular diagonals, AC andBD and holds

$$\|AC\|^2 + \|BD\|^2 = 4$$

then $M^2 = E^2$

Proof

From the previous corollary we have:

$$\|x + y\| \|x - y\| = 2 \tag{8}$$

Also it supposed that

$$\|x - y\|^2 + \|x + y\|^2 = 4$$

From the assumption and (8) easilly follows that $\|x + y\| = \|x - y\|$. Therefore from the theorem (2) we see that $M^2 = E^2$

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