A charecteristic property of a convex centrosymmetric curve

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It is well known, that for a centrosymmetric closed convex curve, the support lines to the endpoints of a chord through the center are parallel see [1]. This property is characteristic of the central convex curves. So we will prove the following theorem.
Theorem: If a closed smooth convex curve (c) in E^2 and an interior point O have the property that (c) possesses parallel supporting lines at the endpoints of every chord through O, then the (c) is centrosymmetrc and O is the center.

Proof
Let O an interior point and $\vec{r} = r(\vec{M})$ the position vector of the point $M \in (c)$.
Suppose $\phi = \angle(\vec{r}, \vec{\epsilon_0})$. We will have:
$\vec{r} = r\vec{r_0}$,hence:

$$\frac{d\vec{r}}{ds} = \vec{\epsilon_0} = \frac{dr}{ds}\vec{r_0} + r\frac{d\vec{r_o}}{ds} \tag{1}$$

or

$$\vec{r}.\vec{\epsilon_0} = r\frac{dr}{ds} \tag{2}$$

that is:

$$\frac{dr}{ds} = cos\phi. \tag{3}$$

The last equation, combining with the well known $ds^2 = r^2(d\theta)^2 + (dr)^2$ gives:

$$\frac{dr(\theta)}{d\theta} = r(\theta)cot\phi. \tag{4}$$

We choose as Ox axis the chord of (c) with middle point the point O. The angle θ is: $\theta = \angle(Ox, OM)$ and $0 \leq \theta \leq 2\pi$.

Let now MN a chord through O and $\vec{OM} = \vec{r}(\theta)$, $\vec{ON} = \vec{r}(\theta + \pi)$ and $\epsilon(\theta)$, $\epsilon(\theta + \pi)$ the parallel support lines at the points M and N respectively. According the above we will have.

$$\frac{1}{r(\theta)}.\frac{dr(\theta)}{d\theta} = \frac{1}{r(\theta+\pi)}.\frac{dr(\theta+\pi)}{d\theta} = cot\phi.$$

or

$$\frac{d(logr(\theta))}{d\theta} = \frac{d(logr(\theta+\pi))}{d\theta}$$

and

$$logr(\theta) = logr(\theta + \pi) + c(const).$$

But $c = 0$ because of $r(0) = r(\pi)$. Therefore follows $r(\theta) = r(\theta + \pi)$.

We point out here that the theorem holds for a general closed convex curve, because if (c) possesses a str.line segment AB and A', B' the central projections of A and B through O respectively on c, then the str.line segment $A'B'$ belongs to c. But, from the neighbor of A (smooth part of c), we see that the points A and A' must be symmetric about O, so the $ABA'B'$ must be parallelogramme. For the case (c)polygon and AB,A'B' opposite sides, we see that AB//A'B' and from the series of the similar triangles with common vertex O, we can see that $\frac{AB}{A'B'} = \frac{A'B'}{AB}$, that is AB=A'B'. Therefore (c) must be a central polygon (an affine image of a regular polygon).

Reference.

1. Theory of convex bodies, T·$Bonnesen, W.Fenchel, B, Associates.$