Two-triangle inequalities.

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Introduction

In this paper our purpose is to reproduce some pioneering inequalities between two triangles, like O.Bottema's (see G.I.1 12.56) and Pedoe- Newberg (see G.I.1 10.8). To this direction we use a compact method obtaining a number of new and very interesting inequalities.

It is well known that from a triangle $T = ABC$ and a point M of its plane we can construct a triangle T' having as sides

$$ax_1, b.x_2, c.x_3$$

We will prove two remarkable lemmas

lemma 1.

We suppose that M is an interior point in the triangle ABC, the triangle T' has angles

$$t_a = \angle BMC - A, \quad t_b = \angle CMA - B, \quad t_c = \angle AMC - C$$

opposite to the sides $a' = ax_1, b' = bx_2, c' = cx_3$ respectively.

Proof

We construct the triangle $AM'C$ similar to AMB. It is elementary to see that the triangles BAC and MAM' are similar and the triangle MCM' has as sides

$$CM, \quad MM' = \frac{a}{c}.AM, \quad CM' = \frac{b}{c}.BM$$

that is the triangle MCM' is similar to the triangle T', with ratio c. Therefore triangle$(a.AM, b.BM, cCM)$ is simimlar to the triangle $(MM', CM", CM)$ and we see $t_a = \angle BMC - A, \quad t_b = \angle CMA - B, \quad t_c = \angle AMC - C$.

Here we have to point out that if M is an exterior point of the triangle ABC we can with a similar way, calculate the angles of the triangle T'. So if M is an exterior pint but in the angle A follows that:

$\angle BMC = A + t_a, \angle CMA = B + t_b, \angle AMB = C + t_c$

lemma 2

Let ABC,$A'B'C'$ two triangles and M the point from the relation

$$x_1 : \frac{a'}{a} = x_2 : \frac{b'}{b} = +x_3 : \frac{c'}{c} = k$$

where $AM = x_1, BM = x_2 = CM = x_3$

We will prove that :

$$k = \frac{abc}{\left(\frac{P}{2} + 8FF'\right)^{1/2}} \tag{1}$$

where $P = \sum a^2(-a'^2 + b'^2 + c'^2)$ and F, F' the area of the triangles ABC and $A'B'C'$. The sum is cyclic over a, b, c and a', b', c'.

proof

The triangle T' with sides ax_1, bx_2, cx_3, according the first lemma, is similar to $A'B'C'$.

Therefore we have:

$$\angle BMC = A + A', \quad \angle AMC = B + B', \quad \angle AMB = C + C'.$$

Cosinus theorem in the triangle BMC gives:

$$a^2 = x_2^2 + x_3^2 - 2x_2x_3cos(A + A')$$

or

$$a^2 = k^2\Big[\frac{b'^2}{b^2} + \frac{c'^2}{c^2} - \frac{2b'c'}{bc}cos(A + A')\Big]$$

From this point, after some easy manipulations, we take (1).

Here we have to point out that the existance of the point M can be assured by the lemma (1). So, if $A + A' \leq 180^0, B + B' \leq 180^0, C + C' \leq 180^0$, the point M is an interior point in the triangle ABC and belongs to the three arcs BC, CA, AB with inscibed angles $A + A', B + B', C + C'$ respectively. Assuming that $A + A' \geq 180^0$ the point M is an exterior point lying in the angle A and so that: $\angle BMC + A = 360^0 - A \geq 180^0$.

That is M is an interior point of the circle ABC.

Also another remarcable comment is that the pedal triangle of the point M is similar to $A'B'C'$.

Indeed we denote by D, E, F the projections of M on BC, CA, AB respectively. We easily find:

$$\angle BMC = A + D$$

but we know that $\angle BMC = A + A'$. Hence $D = A'$. Similarly $E = B', F = C'$.

Inequalities

Bottema's inequality.

Let $ABC, A'B'C'$ two triangles and N be apoint of its plane. We denote $AN = y_1, \quad BN = y_2, \quad CN = y_3$. Bottema's inequality asserts:

$$a'y_1 + b'y_2 + c'y_3 \geq \left[\frac{P}{2} + 8FF'\right]^{\frac{1}{2}} \tag{2}$$

where, we denote

$$P = \sum a^2(-a'^2+b'^2+c'^2) = a^2(-a'^2+b'^2+c'^2)+b^2(a'^2-b'^2+c'^2)+c^2(a'^2+b'^2-c'^2)$$

Proof

We choose the point M so that

$$x_1 : \frac{a'}{a} = x_2 : \frac{b'}{b} = +x_3 : \frac{c'}{c} = k$$

where $AM = x_1, \quad BM = x_2, \quad CM = x_3$

It is known see(G.I.2 XI 3.4) that

$$ax_1y_1 + bx_2y_2 + cx_3y_3 \geq abc$$

Bottema's inequality follows from the above and (1).

Pedoe-Neuberg's inequality

For the two triangles $ABC, A'B'C'$ the Pedoe-Neuberg's inequality asserts:

$$P = \sum a^2(-a'^2 + b'^2 + c'^2) \geq 16FF' \tag{3}$$

where F, F' the area of the triangles.

Proof a.

Let T_0 be the pedal triangle of the point M see Lemma 2 and R_0 its circumradius. The triangle T_0 is similar to $A'B'C'$ so we have $R_0 = kR' \qquad (a)$ where R' the circuradius of $A'B'C'$.

The reader can as problem prove that

$$x_1x_2x_3 \leq R.R_0 \tag{4}$$

From (1) ,(a) and (4) follows the Pedoe-Neuberg's inequality.

Proof b

We obtain a second proof directly, without the help of the relation (4) using only the lemma 2 and the pendal triangle of the point M. We will have:

$$x_1 : \frac{EF}{a} = x_2 : \frac{DF}{b} = x_3 : \frac{FD}{c} = l_1$$

but $EF = x_1 sinA = x_1 \frac{a}{2R}$, therefore

$$l_1 = 2R \tag{5}$$

Also

$$l_1 = \frac{abc}{\left[\frac{P_1}{2} + 8F((DEF)\right]^{\frac{1}{2}}} \tag{6}$$

where $P_1 = \sum a^2(-FE^2 + DE^2 + DF^2)$. From (5),(6) we find

$$\frac{P_1}{2} + 8F.(DEF) = 4F^2$$

but $(DEF) \leq \frac{F}{4}$

Therefore $P_1 \geq 16F(DEF)$.

The similarity of DEF and $A'B'C$" leads us to the Pedoe-Neuberg's inequality.

Other remarcable two triangle inequalities

From the well known inequality

$$ax_1^2 + bx_2^2 + cx_3^2 \geq abc$$

see (G.I.2 3.13), and from lemma (2) formula (1), follows that:

$$\frac{a'^2}{a} + \frac{b'^2}{b} + \frac{c'^2}{c} \geq \frac{P}{2} + 8F.F' \tag{7}$$

From the inequality

$$ax_2x_3 + bx_1x_3 + cx_1x_2 \geq abc$$

and lemma 2,formula (1), follows that:

$$a^2b'c' + b^2a'c' + c^2a'b' \geq \frac{P}{2} + 8F.F' \tag{8}$$

from the well known inequality

$$ax_1 + bx_2 + cx_3 \geq 4F$$

and lemma 2 we take

$$a' + b' + c' \geq \frac{1}{R}\left[\frac{P}{2} + 8F.F'\right]^{\frac{1}{1}} \tag{9}$$

similarly

$$a + b + c \geq \frac{1}{R'}\left[\frac{P}{2} + 8F.F'\right]^{\frac{1}{1}}$$

and finally

$$4S.S' \geq \frac{1}{R.R'}\left[\frac{P}{2} + 8F.F'\right] \qquad (9a)$$

Two interesting triangle inequalities

We construct equilateral triangles BCQ, CAR, ABS on the sides of the triangle ABC. Let M be an interior point in the triangle ABC and we denote $AM = x_1, BM = x_2, CM = x_3$. From the quadralateral $ASMR$ follows:

$$x_1.SR \geq 2(SAM) + 2(RAM)$$

or

$$x_1(b + c) \geq 2(SAM) + 2(RAM)$$

Two other inequalities follows with the same way.So adding we take:

$$(b + c)x_1 + (c + a)x_2 + (a + b)x_3 \geq 2F + 2(BCQ) + 2(ACR) + 2(ABS)$$

or

$$(b + c)x_1 + (c + a)x_2 + (a + b)x_3 \geq 2F + \frac{\sqrt{3}}{2}(a^2 + b^2 + c^2)$$

but we know that $a^2 + b^2 + c^2 \geq 4F\sqrt{3}$, therefore

$$(b + c)x_1 + (c + a)x_2 + (a + b)x_3 \geq 8F \tag{10}$$

The area of the triangle QRS is

$$(QRS) = F + \frac{\sqrt{3}}{4}(a^2 + b^2 + c^2)$$

consequantly

$$(QRS) = F + \frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) \geq 8F$$

For the QRS and the point M holds

$$SR.QM + QR.SM + SQ.RM \geq 4(QRS)$$

so we have

$$\sum (b + c)(x_2 + x_3) \geq 16F \tag{11}$$

Where the sum over $a, b, c \quad x_1, x_2, x_3$
The above (10) and (11) inequalities using the lemma 2 respectively give:

$$\sum (b + c)\frac{a'}{a} \geq \frac{2}{R}\left[\frac{P}{2} + 8F.F'\right]^{\frac{1}{2}} \tag{12}$$

$$\sum (b + c)\left[\frac{b'}{b} + \frac{c'}{c}\right] \geq \frac{4}{R}\left[\frac{P}{2} + 8F.F'\right]^{\frac{1}{2}} \tag{13}$$

Leibniz's formula and its applications to two triangle inequalities.
For the weighted system $A(m_1), B(m_2), C(m_3)$ Leibniz' s formula asserts:

$$m_1x_1^2 + m_2x_2^2 + m_3x_3^2 = mGM^2 + \frac{1}{m}(m_2m_3a^2 + m_3m_1b^2 + m_1m_2c^2) \tag{14}$$

where $m = m_1 + m_2 + m_3$ and G the cendroid of the system, that is:

$$\vec{OG} = \frac{m_1\vec{OA} + m_2\vec{OB} + m_3\vec{OC}}{m}$$

and m_1, m_2, m_3 the distances of the point M from the vertices of the triangle ABC respectively. From (14) arises the inequality:

$$m(m_1x_1^2 + m_2x_2^2 + m_3x_3^2) \geq m_2m_3a^2 + m_3m_1b^2 + m_1m_2c^2 \tag{15}$$

The inequality (15) can produce a long number of inequalities according the position 0f the point G in the triangle ABC. we will give some from these inequalities below. We omit as exercises the proofs.

$$m^2(R^2 - OM^2) = \sum m_1m_2c^2 \tag{16}$$

$$\sum (b + c)x_1^2 \geq \frac{(a + b)(b + c)(c + a)}{4} \tag{17}$$

$$\sum b^2c^2x_1^2 \geq a^2b^2c^2 \tag{18}$$

$$\sum a(b+c)x_1^2 \geq \frac{4S}{3}abc \tag{19}$$

$$\sum bcx_1^2 \geq \frac{2S}{3}abc \tag{20}$$

$$\sum(b^2+c^2)x_1^2 \geq \frac{2}{3}(a^2b^2+b^2c^2+c^2a^2) \tag{21}$$

$$\sum \frac{bc}{b+c}x_1^2 \geq \frac{2abc}{(a+b)(b+c)(c+a)} \tag{22}$$

$$\sum \frac{x_1^2}{b^2+c^2} \geq \frac{1}{2.5} \tag{23}$$

The application of our lemma (2) to the above formulas will give remarcable ineqalities. From (15) we take

$$m\sum m_1^2\frac{a'^2}{a^2} \geq \frac{P/2+8F.F'}{a^2b^2c^2}\sum m_2m_3a^2 \tag{24}$$

The above formula (24) for $A'B'C' = BAC$ gives

$$m\sum m_1\frac{b^2}{a^2} \geq (\sum \frac{1}{a^2})\sum m_2m_3a^2 \tag{25}$$

Formula (25) for $m_1 = m_2 = m_3 = 1$ asserts

$$3(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}) \geq (\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})(a^2+b^2+c^2) \tag{26}$$

Also from (24) for $m_1 = m_2 = m_3$ we take

$$\frac{a'^2}{a^2}+\frac{b'^2}{b^2}+\frac{c'^2}{c^2} \geq \frac{a^2+b^2+c^2}{3}\left[\frac{P/2+8F.F'}{a^2b^2c^2}\right] \tag{27}$$

From (24),for $m_1 = \frac{a}{a'}, m_2 = \frac{b}{b'}, m_3 = \frac{c}{c'}$, we take

$$(\sum \frac{a}{a'})(\sum \frac{a'}{a}) \geq \frac{aa'+bb'+cc'}{abc.a'b'c'}(P/2+8F.F') \tag{28}$$

From (28) for $a' = b, b' = c'c' = a$ we find

$$(a^2c+b^2a+c^2b)(ac^2+b^2a+cb^2) \geq (ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2) \tag{29}$$

From (17) we take :

$$\sum(b+c)\frac{a'^2}{a^2} \geq \frac{(a+b)(b+c)(c+a)}{4a^2b^2c^2}(P/2+8F.F') \tag{30}$$

From (30), setting $a'=b'=c'=1$ we take

$$\sum \frac{b+c}{a^2} \geq \frac{(a+b)(b+c)(c+a)}{4a^2b^2c^2}\Big[(a^2+b^2+c^2)/2+2F\sqrt{3}\Big]$$

or

$$\sum b^2c^2(b+c) \geq (a+b)(b+c)(c+a)F\sqrt{3} \tag{31}$$

because of $a^2+b^2+c^2 \geq 4F\sqrt{3}$ From (18) and lemma (2) we take:

$$\sum \left(\frac{a'bc}{a}\right)^2 \geq P/2+8F.F' \tag{32}$$

and for $a'=b'=c'=1$

$$\sum \left(\frac{bc}{a}\right)^2 \geq \frac{a^2+b^2+c^2}{2}+2F\sqrt{3} \tag{33}$$

From (19),(20),(21),(22),(23) follows respectively.

$$\sum \frac{b+c}{a}a'^2 \geq \frac{4S}{3abc}\Big[P/2+8F.F'\Big] \tag{34}$$

$$\sum \frac{bc}{a^2}a'^2 \geq \frac{2S}{3abc}(P/2+8F.F') \tag{35}$$

$$\sum \frac{b^2+c^2}{a^2}a'^2 \geq \frac{2}{3}\Big[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\Big]\Big[P/2+8F.F'\Big] \tag{36}$$

$$\sum \frac{bc}{b+c}\frac{a'^2}{a^2} \geq \frac{2\Big[P/2+8F.F'\Big]}{abc(a+b)(b+c)(c+a)} \tag{37}$$

$$\sum \frac{a'^2}{a^2(b^2+c^2)} \geq \frac{P/2+8F.F'}{a^2b^2c^2} \tag{38}$$

The above inequalities give very interesting results for the special cases of $A'B'C'=(a',b',c')$. Here we denote by a',b',c' the sides of the ABC. Taking $A'B'C'=(1,1,1)$ or $A'B'C'=(b,c,a)$, or $A'B'C'=(a,b,c)$,we find a class

of very important inequalities.
We assume now that M_1 and M_2 are defined so that—

$$AM_1 : \frac{b'}{a} = BM_1 : \frac{c'}{b} = CM_1 : \frac{a'}{c} = k_1$$

$$AM_2 : \frac{c'}{a} = BM_2 : \frac{a'}{b} = CM_2 : \frac{b'}{c} = k_2$$

where a', b', c' the sides of the triangle $A'B'C'$. According lemma 2 from the inequality (2) follows.

$$\sum bcb'c' \geq \Big[P_1/2 + 8FF'\Big]^{1/2} \Big[P_2/2 + 8FF'\Big]^{1/2} \tag{39}$$

where

$$P_1 = \sum a^2(-b'^2 + c'^2 + a'^2), \qquad P_2 = \sum a^2(-c'^2 + b'^2 + a'^2)$$

Let $ABC, A'B'C'$ two triangles and M' a point. We denote by $A'M' = x'_1, B'M' = x'_2, C'M' = x'_3, a_1 = a'x'_1, b_1 = b'x'_2, c_1 = c'x'_3$, $A_1B_1C_1$ the triangle with sides a_1, b_1, c_1 F_1 its area and $P_1 = \sum a^2(-a_1^2 + b_1^2 + c_1^2)$
According lemma 1 we can find the point M so that

$$AM : \frac{a_1}{a} = BM : \frac{b_1}{b} = CM : \frac{c_1}{c} = k_1 \tag{40}$$

We know that is:

$$a.AM + b.BM + c.CM \geq 4F$$

Hence from lemma 2 and the above inequality we take

$$a'x_1x'_1 + b'x'_2x_2 + c'x'_3x_3 \geq \Big[P_1/2 + 8FF'\Big]^2 \tag{41}$$

From

$$aAM^2 + bBM^2 + cCM^2 \geq abc$$

and (40) we find:

$$\sum bca'^2x_1'^2 \geq P_1/2 + 8FF_1 \tag{42}$$

We see that, the main source of the above 42 inequalities are the lemma

1 and the lemma 2. We found aout and some other inequalities working by similar tehnics, but we think, that the possibilities are clear, so we stop here.

Bibliographical Notes

The recent progress of the Geometrical Inequalities started from 1950 with some excelent Mathematiciens as O.Bottema, D. Mitrinovic, M. Klamkin and others. At that time published the famous book **Geometric Inequalities** by O.Bottema,R.Z. Djordjevic,R.R. Janic, D.S. Mitrinovic,P.M. Vasic. The book is refered in Bibliography as G.I (for us G.I. 1). Twenty years later a second very valuable book published the **Recent Advances in Geometrc Inequalities** by D.S. Mitrinovic, J.E. Pecaric and V. Volonec.We will denote it G.I. 2

The most important part of the Mathematical activity in that part of Mathematics has been published in two journals the **Crux Mathematicorum** from The Canadian Mathematical Society and the **American Mathematical Monthly** from the Mathematical Association of America.