Plane intersections of ellipsoid

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Theorem.

Let (c): $\sum_{i=1}^{n}\frac{x_i^2}{a_i^2} = 1$ be an ellipsoid in E^n and $Q(r) = (M/M.\vec{r_0} = r)$, $Q(0) = (M/M.\vec{r} = 0)$ are the perpendicular n- planes to $\vec{r_0}$ at r and 0 from the center O of (c). We denote by $p_1, p_2, ...p_n$ the directional cosines of $\vec{r_0}$, that is $\vec{r_0} = \frac{\vec{r}}{r} = (p_1, p_2, ...p_n)$ and the intersections $c(0) = Q(0) \cap (c)$, $c(r) = Q(r) \cap (c)$. Then it holds:

$$\frac{area \quad c(r)}{area \quad c(0)} = \frac{\sum_{i=1}^{n} a_i^2 p_i^2 - r^2}{\sum_{i=1}^{n} a_i^2 p_i^2}$$

Proof

The equation of (c) is refered to the Cartesian orthogonal system of axes $Ox_1, Ox_2, ...Ox_n$. We will find the projection of $c(r)$ to the hyperplane $e = Ox_1x_2....x_{n-1}$.

The equation of $Q(r)$ is:

$$Q(r): \qquad p_1x_1 + p_1x_2 + ... + p_nx_n = r \tag{1}$$

So the projection we will be

$$c_e(r): \quad \sum_{i=1}^{n-1}\frac{x_i^2}{a_i^2} + \frac{(r - \sum_1^{n-1} p_i x_i)^2}{a_n^2 p_n^2} = 1 \tag{2}$$

where $c_e(r)$ the projection of $c(r)$ to the hyperplane $e : Ox_1x_2....x_{n-1}$.

from (2) we take:

$$c_e(r) = F(x) = \sum_{i}^{n-1}\left(\frac{1}{a_i^2} + \frac{p_i^2}{a_n^2 p_n^2}\right)x_i^2 + 2\sum_{i>j}^{1,n-1}\frac{p_i p_j x_i x_j}{a_n^2 p_n^2} + 2\frac{r}{a_n^2 p_n^2}\sum_{i}^{n-1} p_i x_i + \frac{r^2}{a_n^2 p_n^2} - 1 = 0$$

Let $\phi(x,x)$ the bilinear form of $F(x)$ and $L(x)$ the linear part of $F(x)$, that is:

$$F(x) = \phi(x,x) + L(x)$$

We denote by d the characteristic determinant of $\phi(x,x)$ and $D(r)$ the characteristic determinant of $F(x)$, $D(0)$ the characteristic determinant of $F(x)$ for $r = 0$.
The calculation of $D(r)$ (quite complicate) gives:

$$D(r) = -\frac{\sum_1^n a_i^2 p_i^2 - r^2}{a_1^2 a_2^2 \ldots . a_n^2 p_n^2} \tag{3}$$

and

$$D(0) = -\frac{\sum_1^n a_i^2 p_i^2}{a_1^2 a_2^2 \ldots . a_n^2 p_n^2} \tag{4}$$

We now suppose that the characteristic roots of the bilinear form $\phi(x,x)$ are: $\lambda_1, \lambda_2, \ldots . \lambda_{n-1}$, so the equation of $c_e(0)$ and $c_e(r)$ are:

$$c_e(0) : \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \ldots . + \lambda_{n-1} x_{n-1}'^2 + \frac{D(0)}{d} = 0$$

$$c_e(0) : \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \ldots . + \lambda_{n-1} x_{n-1}'^2 + \frac{D(r)}{d} = 0$$

Obviously

$$\frac{area \quad c_e(r)}{area \quad c_e(0)} = \frac{area \quad c(r)}{area \quad c(0)},$$

but

$$\frac{area \quad c_e(r)}{area \quad c_e(0)} = \frac{D(r)}{D(0)} = \frac{\sum_1^n a_i^2 p_i^2 - r^2}{\sum_1^n a_i^2}$$

That is

$$\frac{area \quad c(r)}{area \quad c(0)} = \frac{\sum_1^n a_i^2 p_i^2 - r^2}{\sum_1^n a_i^2}$$