Problem

Let $ABCDE$ be a convex pentagon in the plane and $ABC$ the inscribed triangle with the max. area. How big can be the area of the pentagon?. Generalizations.

This is a part of an old problem of mine which is unpublished until now.
The original problem is:
Let $G = A_1, A_2, ...A_m, \quad m \geq d+1$ be a point set in $E^d$ and the volume of every d-simplex with vertices among the $A_i$ is no bigger than $K^d$. Find the the max.volume of the convex cover of $G$
The same problem for $G$ =convex body in $E^d$.
We start the solution for the problem in the plane.
For the quadrilateral $ABCD$ we suppose that the max. inscribed triangle is $ABC$. It is elementary to prove that the max. area for the quadrilateral is the parallelogram $ABCD$ of area $2(ABC)$.
The problem is quite difficult for the convex pentagon $ABCDE$. We suppose that the inscribed triangle of max. area is $ABC$.
We first answer to the following question. Given is the triangle $ABC$. We try to find the points $D, E$ so that the convex pentagon with vertices $A, B, C, D, E$ has the max. area relative to triangle $ABC$.
Let $c = (o, 1)$ be the unit circle and $A_0D_0B_0C_0E_0$ the regular inscribed pentagon. The succession on the circle of the vertices is $A_0, D_0, B_0, C_0, E_0$. Therefore the biggest triangle of the regular pentagon $A_0D_0B_0C_0E_0$ is $A_0B_0C_0$. We now assume that $t$ is the affinity so that :

$$A_0B_0C_0 = t.ABC$$

so, the asked pentagon is

$$ABCDE = t^{-1}.A_0D_0B_0C_0E_0$$

The maximum of the area of the pentagon ABCDE can easily follows from the unique affine $t$ transforming the ABC to $A_0B_0C_0$.
(there is an obvious case, the asked two points $D$ and $E$ be in the triangle $BA'C$ of the parallelograme $ABA'C$. This case does not offer to the problem)

Therefore

$$max.area(ABCDE) = area(ABC).\frac{area(A_0D_0B_0C_0E_0)}{area(A_0B_0C_0)}$$

We work similarly for the original problem. We suppose that the convex cover of $G$ is the polytope $G_1 = A_1A_2...A_p \quad p \geq d+1$ and $S_1$ the $d-simplex$ with the max.volume inscribed in $G_1$. We consider the unit sphere $c = (o.1)$ and we suppose that $B = B_1B_2...B_p$ is the inscribed polytope in $c = (o,1)$ with the max.volume and $S_{o1}$ the simplex with the max.volume inscribed in $B$. We transform by the affinity $t$ , so that:

$$S_{01} = t.S_1$$

We will have:

$$Max.vol.G = max.vol.G_1 = \frac{vol.B}{vol.S_{01}}.vol.S_1$$

We similarly investigate the case G=convex body and $S$ the inscribed simplex with max.volume. Let again $c = (o,1)$ be the unit sphere and $S_0$ the inscribed regular simplex in $c$. By the affinity $t$ we transform

$$S_0 = t.S$$

The max.volume of the convex body $G$ will be:

$$max.vol.G = \frac{vol.c}{vol.S_0}.vol.S$$

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