A characterization of a centrosymmetric convex figure.

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The centrosymmetric figure plays an importand role in convex Geometry. In this paper we discover an interesting theorem about the centrosymmetric figures in the plan.

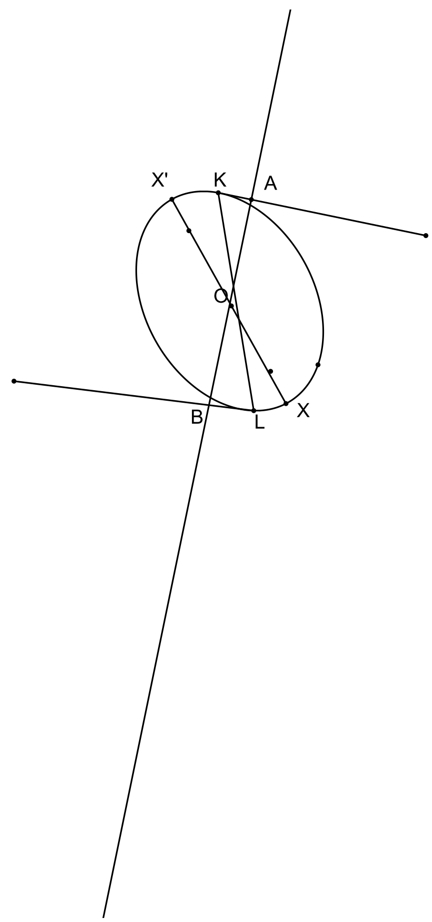
Let (c) be a convex figure of the plane. A diametrical chord AB of (c) parallel to the dierection of the vector $\vec{v}$ is the maximal chord AB of (c) parallel to the vector $\vec{v}$.

Theorem

If every diametrical chord of a convex figure (c) bissects the area of (c), then (c) must be a centrosymmetric figure.

Proof

Let XX' be a diameter of (c) and (O) the middle point of XX'. We choose a Kartesian system of axis xOy, so that $Ox = OX$.



We denote KL a diametrical chord of (c) and $p(\theta)$ the support function of (c) relative to O.
It is well known that the support lines at the points K, L are perallel. Let OA, OB the perpendiculars to the support lines e and h at the points K and L respectively.
We denote $\angle XOAQ = a$, therefore:

$$p(a) = OA, \qquad p(a+\pi) = OB$$

Using elementary Geometry we calculate the area of the triangle KOL. That is:

$$(KOL) = \frac{OA.LB - AK.OB}{2} \tag{1}$$

Taking in our mind that

$$OA = p(a)$$

$$OB = p(a+\pi)$$
$$AK = \dot{p}(a)$$
$$BL = \dot{p}(a+\pi)$$

from (1) we find:

$$(KOL) = \frac{p(a).\dot{p}(a+\pi) - \dot{p}(a).p(a+\pi)}{2} \tag{2}$$

The diametrical chord bissects (c). So we will have:
$Area(curvedKOLXK) + Area(triangleKOL) = Area(curvedKX'LK)$
or, using the formulas of the convex differential we have

$$\frac{1}{2}\int_0^a p(\theta)\rho(\theta)d\theta + areatriagle(KOL) = \frac{1}{2}\int_0^a p(\theta+\pi)\rho(\theta+\pi)d\theta \tag{3}$$

where ρ is the radius of curvature.
From (2),(3) follows

$$\frac{1}{2}\int_0^a p(\theta)\rho(\theta)d\theta + \frac{p(a)\dot{p}(a+\pi) - \dot{p}(a)p(a+\pi}{2} = \frac{1}{2}\int_0^a p(\theta+\pi)\rho(\theta+\pi)d\theta \tag{4}$$

but

$$p(a)\dot{p}(a+\pi) - \dot{p}(a)p(a+\pi) = \int_0^a \Big[p(\theta)\ddot{p}(\theta+\pi) - \ddot{p}(\theta)p(\theta+\pi)\Big]d\theta \tag{5}$$

From (4),(5) follows

$$\frac{1}{2}\int_0^a p(\theta)\rho(\theta)d\theta + \frac{1}{2}\int_0^a \Big[p(\theta)\ddot{p}(\theta+\pi) - \ddot{p}(\theta)p(\theta+\pi)\Big]d\theta = \frac{1}{2}\int_0^a p(\theta+\pi)\rho(\theta+\pi)d\theta \tag{6}$$

This relation (6) holds for every a, so we will have

$$p(\theta)\Big[p(\theta) + \ddot{(}\theta)\Big] + p(\theta)\ddot{p}(\theta+\pi) - p(\theta+\pi)\ddot{p}(\theta) = p(\theta+\pi)\Big[p(\theta+\pi) + \ddot{p}(\theta+\pi)\Big]. \tag{7}$$

It is known that

$$p(\theta) + \ddot{p}(\theta) = \rho(\theta)$$

also

$$p(\theta) + p(\theta+\pi) = B(\theta)$$

so

$$\ddot{p}(\theta) + \ddot{p}(\theta + \pi) = \ddot{B}(\theta)$$

Finally we take

$$\Big[p(\theta) - p(\theta + \pi)\Big]\Big[B(\theta) - \ddot{B}(\theta)\Big] = 0 \tag{8}$$

We can easily see that $B(\theta) \neq \ddot{B}(\theta)$. Because in the opposite case, we will have

$$\int_0^{2\pi} \ddot{B}(\theta) d\theta = \int_0^{2\pi} B(\theta) d\theta$$

but

$$\int_0^{2\pi} \ddot{B}(\theta) d\theta = \dot{B}(2\pi) - \dot{B}(0)$$

Also $\int_0^{2\pi} B(\theta) d\theta = 2L$ where L the perimeter of (c). Therefore from (8) follows that:

$$p(\theta) = p(\theta + \pi)$$

that is according [1], 14-61, (c) must be a centrosymmetric convex figure.

References

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