On the six point property Problem.

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Introduction

P. Erdos asked in [1], whether it is possible for six or seven points in E^3, the triangle with vertices the above points, is acute-angled. Firstly H.T. Croft [2], then B. Grunbaum [3] and finally K. Scutte [4] proved that even six points cannot exist with the above property. In this paper we will expose another proof based on Helly's theorem in a sphere.

Notations

By spherical segment AB on a sphere $(O, R = 1)$ we mean the no greater arc of the Max. circle through the points A, B. The figure F on the sphere $(O, R = 1)$ is convex if and only if does not contain two antipodal points and for every points $A, B \in F$ the spherical segment $AB \in F$.

i. Some preliminary results

a. Helly's Theorem on a sphere.

Helly's theorem on a sphere for a family of convex figures $S = (F_1, F_2, ...F_n)$ on a sphere asserts that if every four convex figures $F_i, i = 1, 2, ..n$ have a common point, then

$$\bigcap_{i=1}^{n} \neq \emptyset$$

We easily can see that the critical number four reduces to three if their diameter $d < \frac{\pi}{2}$. Because some point P remains uncovered by F_i. That is the sphere is equivalent to a plan.

b. lemma.

Let $Q = (A_1, A_2, ...A_n)$ be a point set on a sphere $(o, R = 1)$ with a spherical diameter is $d < \frac{\pi}{2}$. We will prove that Q can be covered by a spherical cap of spherical radius

$$\rho = \frac{35.44}{180}\pi$$

Proof.

Taking in mind that the arcs $A_iA_j, A_jA_k, A_kA_k < \frac{\pi}{2}$ we easily see that

A_i, A_j, A_k belong in a octant of the unit sphere $(O, R = 1)$.(we meant by octant the spherical convex cover of the points A, B, C so that the angles AOB, BOC, COA are equal to $\frac{\pi}{2}$). The spherical cap (T) through A, B, C has a spherical radius

$$\rho = \frac{35.44}{180}\pi$$

that we can easily calculate from some book see, [6] of spherical Trigonometry.
Therefore every three points from Q can be covered by a spherical cap equal to (T). Thus using a well known theorem see[7], which follows from the theorem of Helly, we conclude that Q can be covered by a spherical cap of radius $\rho = \frac{35.44}{180}\pi$.

2. The six point property problem.

Suppose that $P = (A_1, A_2, A_3, A_4, A_5, A_6)$ be a point set in E^3. We will show tha there indixes i, j, k, so that:

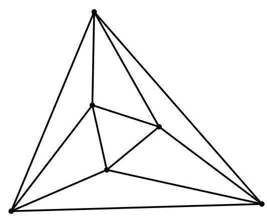
$$\angle A_iA_jA_k \geq \frac{\pi}{2}$$

Proof.

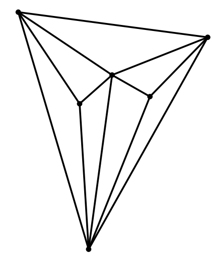
The problem has an obvious solution if some of the points of P bolong to the convex cover of the remaining points. for, if A_6 is in the convex cover of $(A_1, A_2, A_3, A_4, A_5)$ then, there are the numbers $p_i \geq 0$, so that:

$$\sum_{i=1}^{5} p_i.\vec{A_6A_i} = \vec{0} \quad i = 1, 2, 3, 4, 5$$

Squaring the above ralation we easily see that, at least one angle $\angle A_iA_6A_j$ must be obtuse.
Therefore the convex cover of P must be a polyhedron with verices A_i. Also the solution is obvious for the case of quadrelateral or pentagon faces of the P. Hence the polyhedron P has only trigonal faces. Hence we can suppose that A_1, A_2, A_3, A_4 are the vertices of a tetrahedron L. We denote by B_1, B_2, B_3, B_4 the part of the trihedres A_1, A_2, A_3, A_4 without the tetrahedron L. The points A_5, A_6 can be in positions in the B_1, B_2, B_3, B_4 . We can count the trihedres and the dihedres angles for the polyhedron $A_1A_2A_3A_4A_5A_6$ or we can use the Schlegel diagrams of the polyhedron. The Schlegel diagrams are the two types below, also see [3].



Type 1.



Type 2

The Gua's formula

For a polyhedron the Gua's formula is,see [5]:

$$\sum a + n\pi = 2\sum b$$

where n is the number of the vertices, $\sum a$ the sum of the solid angles and $\sum b$ the sum of the dihedre agles. So for the polyhedron P with 6 vertices is

$$\sum a + 6\pi = 2\sum b \tag{1}$$

Also from the elementary Geometry for a n-hedron solid angle the sum of its dihedral angles is:

$$(2n-4)\frac{\pi}{2} < \sum b < 2n.\frac{\pi}{2} \tag{2}$$

The contradiction

Now we assume that for the polyhedron $P = A_1A_2A_3A_4A_5A_6$ holds:

$$A_iA_jA_k < \frac{\pi}{2} \tag{3}$$

We consider the unit sphere $(A_k, R = 1)$, which intersects the semilines $A_kA_i \quad i = 1, 2, 3, 4, 5, 6 \quad i \neq k.$ at the points A_i' Helly's theorem asserts that the respective solid angle A_k must be at most equal to

$$4.\pi.\frac{35.44}{180} = 0.794\pi \tag{4}$$

Therefore for a pyhedron of type 1 we will have:

$$\sum a \leq 6(0.794)\pi,$$

or from (1) follows:

$$\sum b \leq 3.(1.794)\pi \qquad (5)$$

But from (2) we have:

$$2(b_1 + b_2 + ...b_{12}) = 2\sum b > 6(4\frac{\pi}{2}) = 12\pi$$

That is

$$\sum b > 6\pi \qquad (6)$$

From (5) abd (6) we take the asked contradiction.
For a polyhedron of type 2 we work similarly.
Indeed

$$2(b_1 + b_2 + ...b_{12}) > 2.(6\frac{\pi}{2} + 2(4\frac{\pi}{2} + 2(2.\frac{\pi}{2}) = 12\pi$$

Or

$$\sum b > 6\pi$$

which contradicts again to (5) and we have finished.

References

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