Some Inequalities for convex sets

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Abstract

The paper concerns inequalities between fundamental quantities as area, perimeter, diameter and width for convex plane fugures.

Introduction

In this paper we use methods from the Geometry of convex figures and Geometric Inequalities. Both the above mathematical subjects are old but very convenient, fruitful and active in recent times. A classical textbook in the Convexity is the excellent, "Theory of Convex bodies" by T. Bonnesen and W. Fenchel. Another very nice tool is the "Convex Figures" of I. Yaglom and V. Boltyanski. The Yugoslavian and Romanian school, with Mitvinovic and Andrescu, succeded a very interesting theorem in Algebraic and Geometric Inequalities. Here the problem is to find solutions to three interesting inequalities for convex figures in the plane. This problem came to me from a former student of mine, Prof. E. Symeonidis. The problem was published in AXIOMS 2018 7(1) by S.Marcus and F. Nichita. The solution of the first inequality is quite simple but the two others are complex. For the solution I used two lemmas that have particular interest and could be useful to solve other problems.

Problems

Let f be a convex figure in the plane (that is a compact convex set). We denote by G its centroid. D is the maximal chord and d the minimal chord through G. Also L stands for the perimeter, D_F the diameter, d_F the minimal breadth, and A the area of F.

We have to prove:

$$(a) \quad L \geq d\pi.$$

$$(b) \quad d.D > A.$$

$$(c) \quad L.D \geq 4A.$$

Proofs.

Inequality (a).
The formula for the perimeter of a convex figure F see [1],[2] is:

$$L = \frac{1}{2}\int_0^{2\pi} B(\vartheta)d\vartheta$$

where $B(\vartheta)$ is the breadth of F to the direction ϑ, d_F is the min. breadth of F. So we have:

$$L \geq \frac{1}{2}\int_0^{2\pi} d_F.d\vartheta \geq d\pi.$$

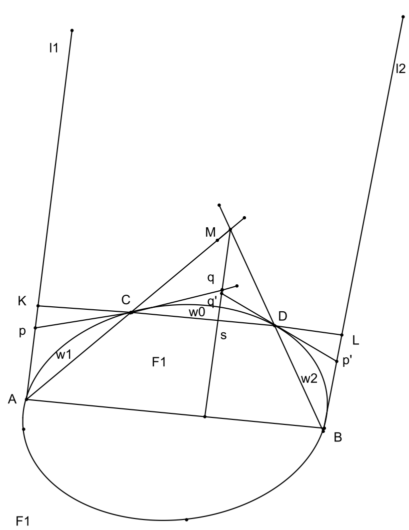
This is because of the obvious $d_F \geq d$
The equality holds for the circle and the convex figures with constant breadth.

For (b) and (c) we need two lemmas.

1. lemma
Let F be a convex figure and AB a diametrical chord. We denote by l_1, l_2 the support lines at the points A, B respectively. We take the chord CD parallel to AB, so that $CD = AB/2$. The str.line CD intersects l_1, l_2 at the points K, L. The chord AB disects F into two parts. We denote the one part by F_1 as in the fig 1 . We will prove that : $area(ABLK) \geq areaF_1$

Proof
Let AC intrsect BD at the point M and MS parallel to AK, The support line at the point C intrsects AK, MS at the points p, q. The support line at the point D intersects BL, MS at the points p', q'. In the triangle AMB the points C and D are the midle points of the sides, so we use equalities of

triangles without the proofs. The triangles pKC, qSC are equal, the same for ApC, MqC, the same for LDp', SDq' and $Mq'D, Bp'D$. We denote w_1, w_2, w_0 the area of the segments of the arcAC,arcBD arcCD.
We have:
$Area(ABLK) = Area(AKC) + Area(ABDC) + Area(BLD)$
$Area(ABLK) = w_1+w_2+w_1'+w_2'+Area(CKp)+Area(DLp')+Area(ABDC) \geq (w_0+w_1+w_2)+Area(ABDC) = AreaF_1$. Because $Area(CKp)+Area(DLp') > w_0$.
By w_1' we denote the area of the triangle ApC with sides $Ap, pC, arcAC$. Analogously w_2'. Finally, we find: $Area(ABLK) > AreaF_1$



2. lemma.
In the perimeter ϑF of the convex set F there are the points A, B, A', B'. The chord AB and $A'B'$ are parallel and the point $P = AA' \cap BB'$ is outside of F. We denote by $arcAB = c, arcA'B' = c'$ on the ϑF and $c \supset c'$, like in the fig.2. We will prove that:

$$\frac{L(c)}{|A-B|} \geq \frac{L(c')}{|A'-B'|}$$

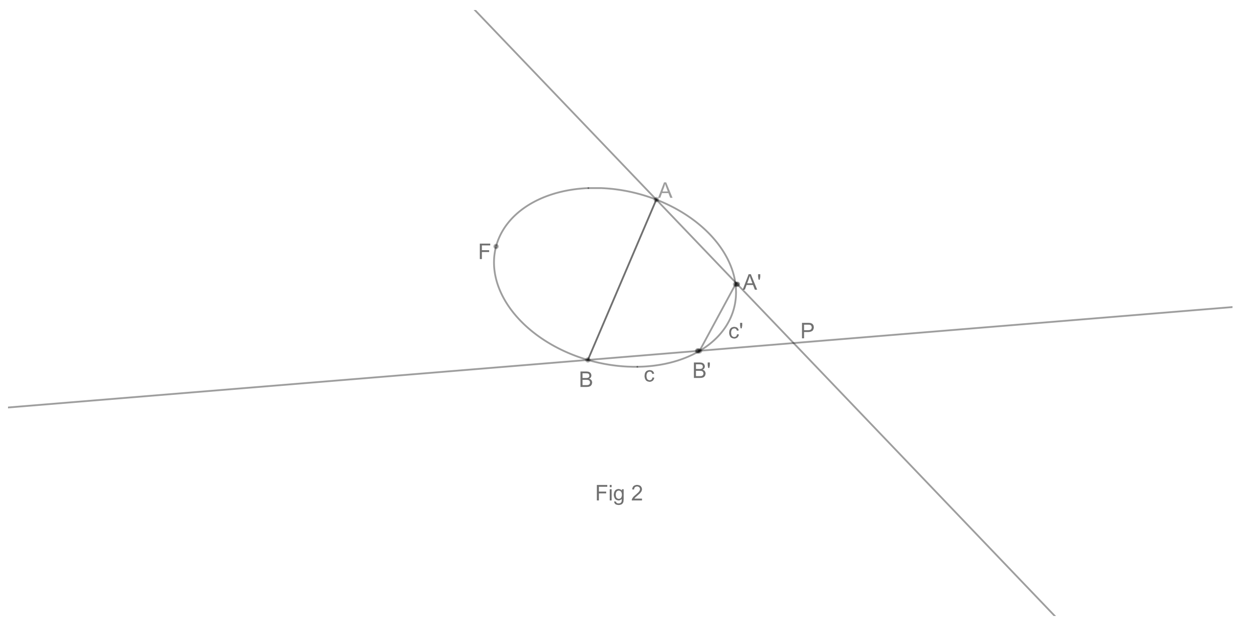


Fig 2

Proof
We denote by M the vector $\vec{O}M$.
We have $A - B = \mu(A' - B')$ We easily find

$$P = \frac{\mu A' - A}{\mu - 1} = \frac{\mu B' - B}{\mu - 1}$$

hence

$$P - A' = \frac{A' - A}{\mu - 1} \qquad (1)$$

and

$$P - B' = \frac{B' - B}{\mu - 1} \qquad (2)$$

but

$$L(c) \geq |A - A'| + |B - B'| + L(c') \qquad (3)$$

From the above (1),(2) follows

$$|A - A'| + |B - B'| = |\mu - 1|(|P - A'| + |P - B'|) \geq |\mu - 1|L(c') \qquad (4)$$

From (3) and (4) we take

$$L(c) \geq |\mu - 1|L(c') + L(c') = \mu L(c')$$

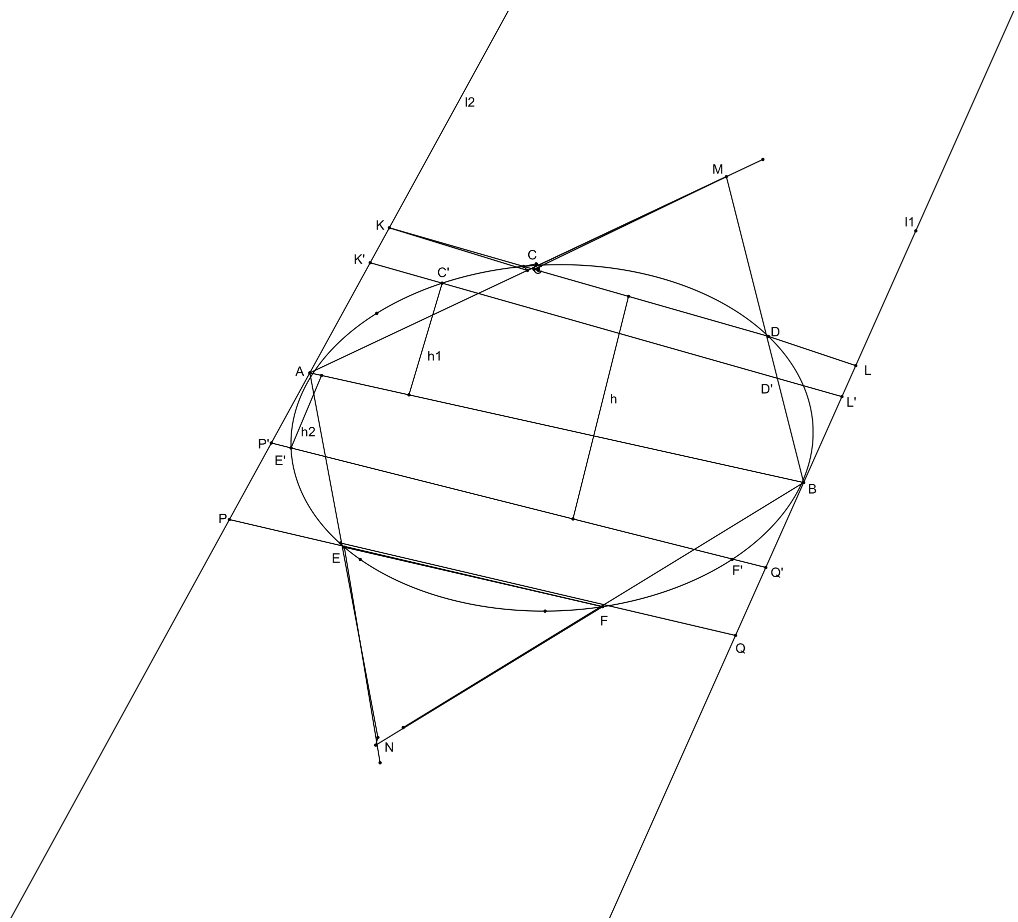
and finally

$$L(c) \geq \frac{|A-B|}{|A'-B'|} L(c')$$

Now the **inequality (b)**.
The continuity of the convexity asserts us that we can choose the diametrical chord AB so that:

$$d_F \leq AB \leq D \leq D_F \qquad (5)$$

Where d_F and D_F stands for the min.breadth and diameter of F respectively.



See fig.3
l_1 and l_2 are the parallel supporting lines at the points A and B

We take on θF the points D, C, E, F so that $\quad CD = EF = \frac{AB}{2}, and \quad CD \| EF \| AB$.
We easily see, according our first lemma that

$$Area(AKC) + Area(BLD) \geq Area(CMD)$$

That means

$$Area(F) < Area(KLQP) = AB.h \qquad (6)$$

where h is the distance between CD, EF.

We now see that the convex F and the orthogonal $PQLK$ have common part $F \cap PQLK$, therefore the position of the centroid G of F depends on the centroid g of the segments $arcCD$ and segment $arcEF$. The point g lies inside the orthogonal $PQLK$. So we conclude that $d \geq h$ and from (6) follows that

$$AreaF < D.d$$

. **inequality c,**
The equality only for F circle. We suppose that F is not a circle.
We translate the str. lines KL, PQ closer towards to AB until to $K'L', P'Q$ such a way the parallelogram $K'L'Q'P'$ has

$$Area(K'L'Q'P') = Area(F) \qquad (7)$$

we have:

$$C'D' > CD, \qquad F'E' > FE$$

where $(C', D') = K'L' \cap F$ and $(F', E)' = P'Q' \cap F$
Let now L_1 the part of the perimeter L over of AB and analogously L_2. We see that $C'D'/AB > 1/2$ and $E'F'/AB > 1/2$. So, from the lemma 2, we easily see that $arcC'D' > L_1/2$ and $arcE'F' > L_2/2$. That is $arcC'D' + arcE'F' > L/2 > arcE'C' + arcD'F'$
Also $AreaF < lengtharcE'C'.AB \quad , AreaF < lengtharcD'F.'AB$ therefore $2AreaF < (lengtharcE'C' + lengtharcD'F')AB$ but $lengtharcE'C' + lengtharcD'F' < lengtharcC'D' + lengtharcE'F'$ hence $4AreaF < (lengtharcE'C' + lengtharcD'F')AB + (lengtharcC'D' + lengtharcE'F)AB < AB.L < D.L$

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