The vector relative to simplex. John's Matrix.

G. A. Tsintsifas
Thessaloniki, Greece

The idea of the vector relative to simplex seems to simplify some theorems and propositions for the Geometry of the simplex.
Let $S = A_1A_2...A_{n+1}$ be a simplex in E^n. It is well known that there exists real positive numbers $c_1, c_2, ...c_{n+1}$ so that:

$$\sum_{i=1}^{n+1} c_i u_i = 0,$$

where u_i the unit vectors perpendiculars to the facets of s and directed to the exterior of S.
We suppose that a is a vector in E^n. We define the vector Da relative to a simplex S as follows:

$$Da = \sum_{i=1}^{n+1} c_i(u_i, a)u_i \qquad (1)$$

The follwing properties can be easily proved.
1. $\forall\lambda \in \Re : \quad \lambda Da = D(\lambda a)$
2. $D(a + b) = Da + Db$
3. $a = 0 \leftrightarrow Da = 0$
4. $(Da, b) = (a, Db) = (Db, a)$
5. $(Da, b_1) + (Da, b_2) = (Da, b_1 + b_2)$
6. $(Da, Db) = \sum_{i,j}^{1,n+1} c_i c_j (u_i, a)(u_j, b)(u_i, u_j)$

John's Matrix

Let $O.x_1x_2...x_n$ be the Cartesian orthogonal system with unit vectors $e_1, e_2, ...e_n$.
The operator $A = \sum_{i=1}^{n+1} c_i u_i \bigotimes u_i$ where :

$$A=\left[\begin{array}{llll}\sum_1^{n+1}c_i(u_i,e_1)^2 & \sum_1^{n+1}c_i(u_i,e_1)(u_i,e_2) & \cdot & \sum_1^{n+1}c_i(u_i,e_1)(u_i,e_n)\\ \sum_1^{n+1}c_i(u_i,e_2)(u_i,e_1) & \sum_1^{n+1}c_i(u_i,e_2)^2 & \cdots & \sum_1^{n+1}c_i(u_i,e_2)(u_i,e_n)\\ \multicolumn{4}{l}{\dotfill}\\ \multicolumn{4}{l}{\dotfill}\\ \sum_1^{n+1}c_i(u_i,e_n)(u_i,e_1) & \sum_1^{n+1}c_i(u_i,e_n)(u_i,e_2) & \cdots & \sum_1^{n+1}c_i(u_i,e_n)^2\end{array}\right] \quad (2)$$

is the well known John's matrix.
Introducing in A the notation of a vector relative to simplex we will have:

$$A=\left[\begin{array}{llll}(De_1,e_1) & (De_1,e_2) & \cdots & (De_1,e_n)\\ (De_2,e_1) & (De_2,e_2) & \cdots & (De_2,e_n)\\ \multicolumn{4}{l}{\dotfill}\\ \multicolumn{4}{l}{\dotfill}\\ (De_n,e_1) & (De_n,e_2) & \cdots & (De_n,e_n)\end{array}\right] \qquad (3)$$

Let now the vector $v=\sum_i^n x_ie_i$ and $|v|=1$, then we will have:

$$(Dv,v)=\sum_{i,j}^{1,n} x_ix_j(De_i,e_j) \qquad (4)$$

Therefore we see that

$$(Dv,v)=\sum_{i=1}^{n+1} c_i(u_i,v)^2=\sum_{i,j}^{1,n} x_ix_j(De_i,e_j) \qquad (3a)$$

setting also $a_{ij}=(De_i,e_j)$ and denoting $\lambda_1\geq\lambda_2\geq...\geq\lambda_n$ the characteristic roots of the matrix $[a_{ij}]$, we take:

$$max(Dv,v)=\lambda_1 \qquad 3(b)$$

$$min(Dv,v)=\lambda_n. \qquad 3(c)$$

Let now $b_{ij}=(De_i,De_j)$. We can see that:

$$b_{ij}=\sum_{k=1}^{n} a_{ik}a_{jk},$$

that is $B=[b_{ij}]=A\cdot A^T=A^2$.
Hence the roots of B are $\lambda_1^2,\lambda_2^2,...\lambda_n^2$.
Conclusion:

$$\lambda_1^2\geq(Dv,Dv)\geq\lambda_n^2$$

The regular Simplex S_0

The following propositions are easy .

1. $(u_i, u_j) = -\frac{1}{n}$ for $i \neq j$, $\qquad c_i = \frac{n}{n+1}$

2. For the Cartesian system of coordinates $e_1, e_2, .. \quad ..e_n$, we set $e_i = l_1 u_1 + l_2 u_2 + ... + l_{n+1} u_{n+1}$ where the l_i are real numbers, the barycentric coordinates of e_i. that is $l_1 + l_2 + ...l_{n+1} = 1$

We easily find that

$$l_k = \frac{1 + n(e_i + u_k)}{n+1} \tag{5}$$

Also from $e_i^2 = 1$ that is $(\sum l_i)^2 = 1$, we take that $\sum l_i^2 = 1$ Squaring (4) and summing we take

$$\sum_{k=1}^{n+1} (u_k, e_i)^2 = \frac{n+1}{n} \tag{6}$$

Similarly working with $e_j = l_1' u_1 + ...l_{n+1}' u_{n+1}$ we find $\sum l_i l_i' = \frac{1}{n+1}$ and then

$$\sum_{k=1}^{n+1} (e_i, u_k)(e_j, u_k) = \frac{(n+1)^2 \sum l_i l_i' - (n+1) \sum l_i' - (n+1) \sum l_i + n + 1}{n^2} = 0 \tag{7}$$

3. The John' s Matrix for the regular simplex is $A_0 = I_n$. This follows from (2) and (6),(7).

Prblem

The $B(v_0)^{n-1}$ sphere inscribed in a n-simplex

Let p_v be a plane of direction v and S is a n-simplex in E^n. We denote by $S(v) = T(p_v) \cap S$. The inscribed sphere in $S(v)$ is $B(v)^{n-1}$. We suppose that $B_0(v)^{n-1} = Max.B(v)^{n-1}$ for every $S(v)$. We call the inscribed sphere $B(v_0)^{n-1}$ in the n- simplex the Min.$B_0(v)^{n-1}$ for all the vectors $v_0 \in E^n$

To understood the problem it is better to see it in the E^3. Suppose that $ABCD$ is a regular tetrahedron and p a plane of direction p. The translation of p intersects from $ABCD$ a triangle T or quadrilateral Q. We consider the incircle C in T or Q. The incircle C a has for some position MaxC=C_0. We are looking for the Min of C_0 for every direction of p in E^3.Another formulation of the same circle is the following: We are looking for the max. circle who can fit inside of a tetrahedron in every direction. As we have prove this circle for the regular tetrahedron is the inscribed circle, of radius $\sqrt{3}/\sqrt{2}$ in the in the square with vertices the four middlepoints, of the four opposite

edges of the regular tetrahedron with radius equal to one. Point out that in this tetrahedron the inradius of a face is $\sqrt{2}$.
Let $S = A_1A_2....A_{n+1}$ a n-simplex in E^n and p_v a $n-1$ plane. We set

$$S(v) = p_v \cap S$$

(K, r) is the inscribed B^{n-1} sphere in $S(v)$. Our first problem is to determine the Max.r for all the tranlates of p_v, with direction v.
We introduce a Cartesian Coordinates system $Oe_1e_2....e_n$ where $e_n = v$ that is e_n normal to p_v. We asumme $u_1, u_2, ...u_{n+1}$ the unit vectors perpendicular to the faces. As we know wecan find the positive real numbers $c_1, c_2, ...c_{n+1}$ so that:

$$\sum_{i=1}^{n+1} c_i u_i = 0,$$

$$A = \sum_{n=1}^{n+1} c_i u_i \otimes u_i \neq I_n$$

We denote by y the vector of position of the center K of the inscribed sphere B^{n-1} in $S(v)$ and r the radius. For every point $x \in (K, r)/|x - y| = r$ holds:

$$\sum_{j=1}^{n-1} (x - y, e_j)^2 = r^2 \tag{8}$$

The point

$$x_i = y + \sum_{j=1}^{n-1} r(u_i, e_j)e_j.\Big[\sum_{j=1}^{n-1} (u_i, e_j)^2\Big]^{-\frac{1}{2}} \tag{9}$$

is on the sphere (K, r) and on a facet of S. See fig 1.

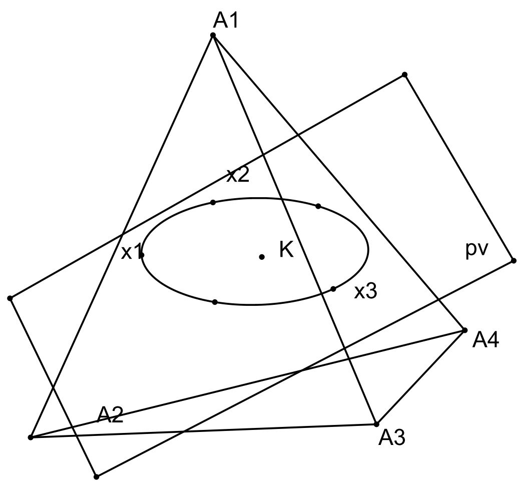

fig 1

The vector u_i is projected to the (n-1)-plane p_v to the vector

$$P_{n-1}u_i = \sum_{j=1}^{n-1}(u_i, e_j)e_j \tag{10}$$

We denote by P_k the projection to the k- plane.We work only for k=n-1 so for smplicity the n-1 projection we will denote by P .See fig 2

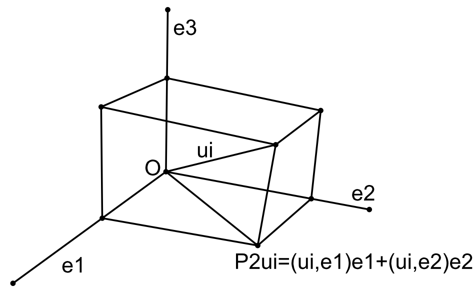

fig 2

From (9),(10) we have:

$$x_i = y + \frac{Pu_i}{|Pu_i|} \tag{11}$$

that is because the point x_i lies to the sphere (K, r) and to a facet of the simplex S. Also for the non exterior points x of the simple S holds

$$(x, u_i) \leq 1$$

The equality for the points of (K, r) on the facets of the simplex. Hence,

$$\left(y + r\frac{Pu_i}{|Pu_i|}, u_i\right) \leq 1$$

and for the points x_i of (k, r) on the facets holds:

$$(y, u_i) + r\frac{(Pu_i, u_i)}{|Pu_i|} = 1 \tag{12}$$

or

$$(y, u_i) + r|Pu_i| = 1$$

for $i = 1, 2, 3....n + 1$, so we have

$$y.\left(\sum_{i=1}^{n+1} c_i u_i\right) + r\sum_{i=1}^{n+1} c_i|Pu_i| = \sum_{i=1}^{n+1} c_i$$

We suppose that $e_n = v$ is at the moment constand, we see that the Max.$r = r_1$, will be:

$$r_1 = \frac{\sum_{i=1}^{n+1} c_i}{\sum_{i=1}^{n+1} c_i |Pu_i|} \tag{13}$$

The last part of our problem now is to find the r_0=Min.r_1,for every vector $v \in E^3$
We see that

$$\sum_{j=1}^{n} (u_i e_j) = (u_i)^2 = 1$$

Or

$$|Pu_i|^2 + (u_i, v)^2 = 1 \tag{14}$$

for $e_n = v$ and

$$c_i^2 |Pu_i|^2 + c_i^2 (u_i, v)^2 = c_i^2$$

Therefore

$$\sum_{i=1}^{n+1} c_i^2 |Pu_i|^2 = \sum_{i=1}^{n+1} \Big[c_i^2 - c_i^2 (u_i v)^2 \Big] \tag{15}$$

lFrom Cauchy-Schwarz inequality follows that

$$(\sum_{i=1}^{n+1} c_i) \Big[\sum_{i=1}^{n+1} c_i |Pu_i|^2 \Big] \geq \Big[\sum_{i=1}^{n+1} c_i |Pu_i| \Big]^2 \tag{16}$$

From (14) and (16) follows that

$$(\sum_{i=1}^{n+1} c_i) \Big[\sum_{i=1}^{n+1} c_i - \sum_{i=1}^{n+1} c_i (u_i v)^2 \Big] \geq \Big[\sum_{i=1}^{n+1} c_i |Pu_i| \Big]^2$$

Hence from the above , (3c) and (16) follows that r_0=Min.r_1 where

$$r_1 = \frac{\sum_{i=2}^{n+1} c_i}{\sum_{i=1}^{n+1} c_i |Pu_i|} \geq \frac{\sum_{i=1}^{n+1} c_i}{\sqrt{\sum_{i=1}^{n+1} c_i (\sum_{i=1}^{n+1} c_i - \lambda_n)}} = \frac{\sqrt{\sum_{i=1}^{n+1} c_i}}{\sqrt{\sum_{i=1}^{n+1} c_i - \lambda_n}} \tag{17}$$

Where λ_n the min. charecteristique root of the John's Matrx, see(3c)
The equality for $|c_i(u_, iv)| = |c_j(u_, jv)|$. That is possible for n=odd. For n=even in the above formula is true only the inequality. For the equilateral triangle the equality is true for $|(u_1, v)| = |(u_2, v)| = 1/2$ and $|(u_3, v)| = 1$.

1.John's theorem for the simplex.

The Eucledean ball is the ellipsoid of maximum volume contained in the simplex $S = A_1A_2...A_{n+1}$ if and only if:
(a).$\sum_1^{n+1} c_i u_i = 0$
(b).$\sum_1^{n+1} c_i u_i \bigotimes u_i = I_n$ the identity matrix.
The u_i and c_i as already defined.
2.For the inscribed sphere (I, r) in the simplex S holds:

$$r \leq \frac{\sum_{i=1}^{n+1} c_i}{n\lambda_n} \tag{18}$$

where ,$\lambda_1, \lambda_2, ...\lambda_{n-1}\lambda_n$ the charectiristic roots of the John's matrics
Proof

Let $S = A_1A_2....A_{n+1}$ simplex in E^n and (I, r) is the inscribed sphere,$\lambda_1 \geq, \lambda_2 \geq, \lambda_3....\lambda_{n-1} \geq \lambda_n$ the charectiristic roots and $O.e_1e_2.....e_n$ one orthonormal system of reference and y the vector of I. we will have

$$\sum_j^{1,n}(x - y, e_j)^2 \leq 1 \tag{19}$$

for every point $x \in (I, r)$

As usually we denote by $u_1, u_2, ...u_{n+1}$ the unit vectors normal to the edges directed to the outside of S. the point

$$x_i = y + \frac{r.\sum_{j=1}^{n}(u_i, e_j)e_j}{\left[\sum_{n=1}^{n}(u_i, e_j)^2\right]^2} \tag{20}$$

is on the sphere (I, r) and on a face of S.
From (19),(20), summing with respect i we take:

$$r.\sum_{i=1}^{n+1} c_i\Big[\sum_{j=1}^{n}(u_i, e_j)^2\Big]^{\frac{1}{2}} \leq \sum_{i=1}^{n+1} c_i \tag{21}$$

where $\sum_{i=1}^{n+1} c_i u_i = 0$. But $\sum_{j=1}^{n+1}(u_i, e_j)^2 = u_i^2 = 1$. So from this and the above we can have:

$$r.\sum_{i=1}^{n+1} c_i\Big[\sum_{j=1}^{n}(u_i, e_j)^2\Big]^{\frac{1}{2}}\Big[\sum_{j=1}^{n}(u_i, e_j)^2\Big]^{\frac{1}{2}} \leq \sum_{i=1}^{n+1} c_i$$

or,

$$r.\sum_{i=1}^{n+1} c_i \sum_{j=1}^{n} (u_i, e_j)^2 \leq \sum_{i=1}^{n+1} c_i$$

or

$$r.\sum_{j=1}^{n} . \sum_{i=1}^{n+1} c_i (u_i, e_j)^2 \leq \sum_{i=1}^{n+1} c_i$$

but it is known from (3c) that

$$\sum_{i=1}^{n+1} c_i (u_i, e_j)^2 \leq \lambda_n$$

and finally

$$r \leq \frac{\sum_{i=1}^{n+1} c_i}{n\lambda_n}$$

3.Let

$$s_1 = \sum_{1}^{n} \frac{x_i^2}{a_i^2} - 1$$

be the inscribed ellipsoid in simplex S, then

$$\sum_{j=1}^{n} a_j \leq \frac{\sum_{i=1}^{n+1} c_i}{\lambda_n} \tag{22}$$

For an orthonormal system in E^n and every $x \in s_1$ holds

$$\sum_{j=1}^{n} a_j^{-2} (x - y, e_j)^2 \leq 1$$

where y the center of the s_1. Let $x_i \in s_1 \cap S$ then

$$x_i = y + \Big(\sum_{j=1}^{n} a_j^2 (u_i e_j) e_j \Big) \Big(\sum_{j=1}^{n} a_j^2 (u_i e_j)^2 \Big)^{\frac{-1}{2}}$$

From the above we see that

$$\sum_{i=1}^{n+1} c_i \Big(\sum_{j=1}^{n} a_j^2 (u_i, e_j)^2 \Big)^{\frac{1}{2}} \leq \sum_{i=1}^{n+1}$$

and from Cauchy-Schwarz inequality follows

$$\sum_{j=1}^{n} a_j(u_i, e_j)^2 \leq \Big(\sum_{j=1}^{n} a_j^2 (u_i, e_j)^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{n} (u_i, e_j)^2\Big)^{\frac{1}{2}}$$

but

$$\sum_{j=1}^{n+1} (u_i e_j)^2 = 1$$

hence

$$\sum_{i=1}^{n+1} c_i . \sum_{j=1}^{n} a_j (u_i, e_j)^2 \leq \sum_{i=1}^{n+1} c_i$$

or

$$\sum_{j=1}^{n+1} a_j . \sum_{i=1}^{n+1} c_i (u_i, e_j)^2 \leq \sum_{i=1}^{n+1} c_i$$

and according (3c) we take

$$(\sum_{j=1}^{n} a_j)\lambda_n \leq \sum_{i=1}^{n+1} c_i$$

4. Some other relations

$$0 \leq \sum_{i=1}^{n+1} c_i (1 - (x, u_i))(R + (x, u_i))$$

where $|x| = R$. From the above follows that:

$$0 \leq R. \sum_{i=1}^{n+1} c_i + (1 - R) \sum_{i=1}^{n+1} c_i (x, u_i) - \sum_{i=1}^{n+1} c_i (x, u_i)^2$$

But

$$\sum_{i=1}^{n+1} c_i (x, u_i)^2 \geq |x|^2 \lambda_n$$

that is because of $(1 - R)\sum_{i=1}^{n+1} c_i (x, u_i) = 0$
Therefore

$$R \leq \frac{\sum_{i=1}^{n+1} c_i}{\lambda_n}.$$

5.Some interesting results about the John's matrix.
Let

$$A = [a_{ij}] = (De_i, e_j) = \sum_{k=1}^{n+1} c_k(u_k, e_j(u_k, e_j)$$

The characteristic equation is

$$|A - \lambda.I| = 0$$

that is:

$$p_0\lambda^n + p_1\lambda^{n-1} + ... + p_n = 0$$

So $p_0 = 1, \quad p_1 = trA = -(a_{11} + a_{22} + ... + a_{nn})$ hence

$$\lambda_1 + \lambda_2 + ...\lambda_n = \sum_{i=1,n+1} \sum_{j=1,n} c_(u_i, e_j)^2 = \sum_{i=1}^{n+1} c_i$$

where $\lambda_1 \geq \lambda_2.... \geq \lambda_n$ the characteristic roots.
Obviously

$$n\lambda_1 \geq \sum_1^{n+1} c_i \geq n\lambda_n$$

For $A = I_n$, we have $\lambda_i = 1$

6. The proof of John Theorem for the simplex.

The condition of John

$$\left[\sum_{i=1}^{n+1} c_i u_i = 0 \quad , A = I_n\right] \tag{23}$$

We suppose that (23) is true. We will prove that the max inscribed ellipsoid in a simplex will be sphere.
From the above (3) the condition (23) and the Cauchy-Schwarz Theorem we see that

$$a_1 a_2 ....a_n \leq \frac{1}{\lambda_n^n}$$

but from (23) $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$. So $a_1 = a_2 = ... = a_n$.
We suppose now that the max. ellipsoid is sphere, then we will prove (23).
Indeedl, according the well known Day's see (4) theorem follows that the simplex will be regular so the condition of John that is (23) is true

7. The Algebra of the vector relative to a simplex

In the beginning of this paper, we started with the basic definitions of the

vector relative to a simplex. Combining Linear Algebra and Geometry provides sometimes interesting results. In the following we will mention some properties of this special vectors without giving the proofs.
1.$D(a, Da) = \sum_{i,j} c_i c_j (u_i, u_j)(u_i, a)(u_j, a)$
2.$(Da, DB) = (D(a), b)$
3.$(D(a_1 + a_2), b) = (D(Da), b)$
4.$(Da, b) = 0$
5.$-(Da, b) = (D(-a), b) = (Da, -b)$
6$(D(Dv), v) - (Dv, v) = (D(Dv - v), v)$
7$(De_i, De_j) = \sum_{k,p} c_k c_p (u_k, e_i)(u_p, e_j)(u_k, u_p)$

8. $\quad (Dv_1, v_2) = \lambda_1(v_1, v_2) \bigwedge (Dv_2, v_1) = \lambda_2(v_1, v_2) \rightarrow \lambda_1 = \lambda_2$

9.$|Da|^2 = (Da, Da)$
10. $(Da, a) \leq |Da||a|$
11.$(Da, Db) \leq |Da|Db|$
12.$(Da, b)^4 \leq (Da, Da)(Db, Db)|a|^2|b|^2$

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