Problems of plane Geometry.

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Problem1

Let ABC be a triangle and the points A', B', C' are on the sides BC, CA, AB respectively so that $BA' = CB' = AC'$.
Prove that the incenters of the triangles $ABC, A'B'C'$ are common, if and only if, the triangle ABC is equilateral.
Solution
We consider the common incenter O as the origin of the Cartesian system. We will denote on the plane of ABC the vector $\overrightarrow{OM}$ by M. The sides of the triangles ABC and $A'B'C'$ are a, b, c and a', b', c'.
We know that the incenter of ABC is the point
$\frac{aA+bB=cC}{a+b+c}$. Therefore we will have.

$$\frac{aA + bB + cC}{a + b + c} = O \qquad \frac{a'A' + b'B' + c'C'}{a' + b' + c'} = O \tag{1}$$

Let us suppose $BA' = CB' = AC' = p$. We easily see that

$$A' = \frac{pC + (a - p)B}{a}, \qquad B' = \frac{pA + (b - p)C}{b}, \qquad C' = \frac{pB + (c - p)}{c} \tag{2}$$

Let G the barycenter of the triangle $A'B'C'$, it is:

$$A' + B' + C' = 3G \tag{3}$$

We can easily see that: In a triangle, if the centroid coincides with the incenter, then the triangle must be equilateral.
That is if $G = O$ then the triangle $A'B'C'$ is equilateral. If $G \quad is \quad not \quad O$ then the triangle $A'B'C'$ is not equilateral.

From the above we conclude that the condition to be the triangle $A'B'C'$ equilateral is:

$$A' + B' + C' = 0$$

hence from (2) have

$$abc(A'+B'+C') = pbcC+cb(a-p)B+pacA+ac(b-p)C+pabB+ab(c-p)A = 0 \tag{4}$$

From (1) we have $A = -\frac{bB+cC}{a}$. We replace A in (4) and we have after some Algebra

$$(bc(a-p)-pbc+abp-b^2(c-p))B+(pbc-pc^2+ac(b-p)-bc(c-p))C = 0 \tag{5}$$

Relation (5) holds only for when coefficients of the vectors B and C are zero. That is

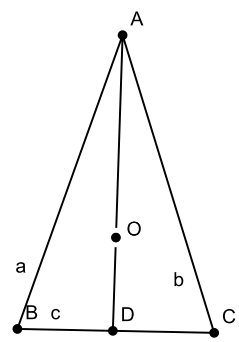
$$bc(a-p) - pbc + pab - b^2(c-p) = 0 \tag{6}$$

$$pbc - pc^2 + ac(b-p) - bc(c-p) = 0 \tag{7}$$

From (6) and (7) we take

$$(a-b)(b-c)(c-a) = 0 \tag{8}$$

From (8) follows that the triangle ABC is only isosceles. Let us suppose that $b = c$.That is for the triangle ABC is $AB = AC$



aFrom(1) is: $aA + bB + bC = 0$ that is $aA + b(2D) = 0$ where AM is the altitude of the isosceles triangle BAC. But $2.|\vec{OD}| = |\vec{OA}|$ or a=b. that is b=c and b=a hence the triangle ABC is equilateral

comments

For the solution of the following problems we are using the same method

Problem 2

The same us the problem 1 but here the centroids of the two triangles coincide.
So here we have

$$A + B + C = 0 \qquad A' + B' + C' = 0$$

We replace in (4) $A = -B - C$. The vectors B, C are independent so their coefficients are zero.Finally we have

$$bc + ab = 2ac \qquad bc + ac = 2ab$$

And from here $a = b = c$

problem 3

The triangle ABC is equilateral, $A'B'C'$ is the inscribed triangle. Here the incenters coincide. We have to prove that the triangle $A'B'C'$ is equilateral. Another problem of the same type is the following. The inscribed triangle $A'B'C'$ in ABC is equilateral. The incenters coincide. We prove that the triangle ABC is equilateral. For the solution we can use the same method. But here there is also another one simple solution.

Problem 4

The inequality of Pedoe
Let ABC and $A'B'C'$ two triangles. We denote by a, b, c and a', b', c' their sides and E, E' the area. We call the quantity of Pedoe the symmetrc (for a, b, c, a', b', c') formula

$$P = \sum a^2(-a'^2 + b'^2 + c'^2)$$

The inequality of Pedoe is:

$$P \geq 16EE' \tag{9}$$

For the proof we will use the Leibnitz s formula

$$\sum_1^n m_i.pA_i^2 = m.pG^2 + \frac{1}{m}\sum_{i>j} m_i m_j.a_{ij}^2 \tag{10}$$

where $A = (A_1, A_2, ...A_n)$ is an simplex, $G = (m_1, m_2, ...m_n)$ the centroid, that is

$G = \frac{\sum m_i A_i}{\sum m_i}$. and p $\in R^n$ and $m = \sum m_i$.

For the triangles ABC and $A'B'C'$, we suppose

$$G(m_1, m_2, m_3) = G(a^2(-a'^2 + b^2 + c^2), b^2(a^2 - b^2 + c^2), c^2(a^2 + b^2 - c^2))$$

and p=O the circumcenter of ABC. We will have

$$R^2\Big[\sum a^2(-a'^2 + b;^2 + c'^2)\Big]^2 =$$

$$= \sum a^2b^2c^2(a'^2 - b'^2 + c'^2)(a^2 + b'^2 - c^2) =$$

$$= a^2b^2c^2\Big[\sum a^4 - (b'^2 - c'^2)^2\Big]^2 =$$

or

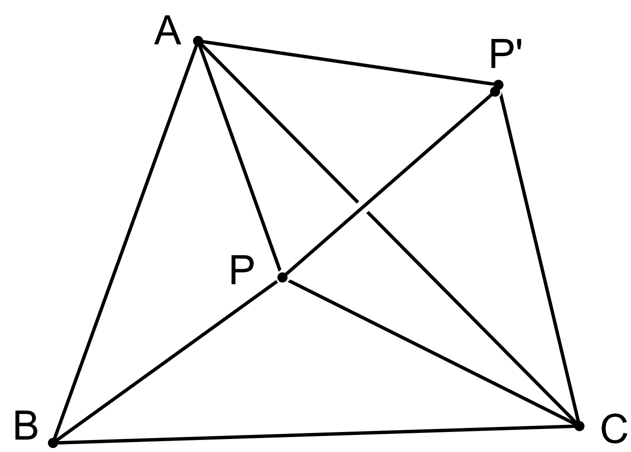
$$= R^2\Big[\sum a^2(-a'^2 + b'^2 + c'^2)\Big]^2 \geq a^2b^2c^2 16E^2$$

or

$$\sum a^2(-a'^2 + b^2 + c'^2) \geq \frac{4ER.4E'}{R} = 16EE'$$

problem 5

Let ABC be atriangle and P an interior point. We easily prove that there is a triangle wit sides aAP, bBP, cCP. where by AP, BP, CP we denote the str. line segments AP, BP, CP. We construct the triangle APP' similar to APB



The triangles APP' and ABC are similar. The triangle $PP'C$ has sides $PP' = \frac{a}{c}BP$ and $P'C = \frac{b}{c}$

The triangles (aAP, bBP, cCP) and $c.(PP'C)$ are similar.

The angles are

$\angle PC - \angle A = \angle PCP'$ opposite of aAP

$\angle CPA - \angle B = \angle CPP'$ opposite of bBP

$\angle APB - \angle C = \angle PP'C$ opposite of cCP

A second triangle $A'B'C'$ has sides a', b', c' . we determine the point P in the triangle ABC so that:

$$\frac{aAP}{a'} = \frac{bBP}{b'} = \frac{cCP}{c'} = m \tag{11}$$

Cos. theorem in the triangle BPC gives:

$$a^2 = m^2\Big[\frac{b'^2}{b^2} + \frac{c'^2}{c^2} - \frac{2b'c'}{bc}cos(A + A')]$$

After some Algebra we take

$$a^2 = \frac{m^2}{b^2c^2}\Big[b'^2c^2 + c'^2b^2 - \frac{(b^2 + c^2 - a^2)(b'^2 + c'^2 - a'^2)}{2} + 8EE'\Big]$$

Consequently

$$a^2 = \frac{m^2}{b^2c^2}\Big[\frac{P}{2} + 8EE'\Big]$$

where P is the quantity Pedoe. and

$$m = \frac{abc}{\Big[\frac{P}{2} + 8EE'\Big]^{1/2}} \tag{12}$$

From the above we can find new triangle inequalities of Pedoe s style. For example taking in mind the inequality $aa' + bb' + cc' \geq 4\sqrt{3EE'}$ we easily find that

$$\sum a^2AP \geq \frac{4abc}{P^{1/2}}\sqrt{3EE'}$$

Here we can prove anothere important inequality. The inequality of O. Bottema. that is for two triangles ABC and $A'B'C'$ and the internal points P andP' holds

$$a'AP + b'BP + c'CP \geq (P/2 + 8EE')^{1/2}$$

We need n elementary inequality, for the triangle ABC and two internal points P and M we have

$$a.AP.AM + b.BP.BM + C.CP.CM \geq abc$$

From (11) and (12 and the above) follows Bottema s inequality

the Apollonius circles, the isodynamic centers

It is known that the Apollonius circles have a common chord PP' . The points P, P' are called Isodynamic points. One of the two points let the P is interior in the triangle ABC. We easily see that

$$\frac{AP}{bc} = \frac{BP}{ac} = \frac{CP}{ab}$$

so we can take

$$\frac{aAP}{abc} = \frac{bBP}{abc} = \frac{cCP}{abc}$$

Hence the triangle (aAP, bBP, cCP) is equilateral . Therefore we have:

$$\angle BPC = 60^0 + A \quad \angle CPA = 60^0 + B \quad \angle APB = 60^0 + C.$$

we also can prove that the feeds of P on the sides of ABC are the vertices of equilateral triangle, the minimum inscribed trianle in the ABC.

Referenses

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