A problem of Geometry.

G. A. Tsintsifas
Thessaloniki, Greece

Let $A_1A_2A_3$ be an equilateral triangle inscribed in the circle $(O, R = 1)$. The point M is on the circle so $\vec{OM} = v, |v| = 1$. We denote $\vec{OA_i} = u_i$.
(a). Prove: $\sum_1^3 (u_i, v)^2 = 3/2$
(b) Find Max $\sum_1^3 |(u_i, v)|$

Solution

(a)
By symmetry

$$u_1 + u_2 + u_3 = 0$$

Also

$$u_1^2 + u_1.u_2 + u_1.u_3 = 0$$

That is

$$1 + 2u_1.u_2 = 0$$

We easily see that

$$(u_i.u_j) = -\frac{1}{2}, for \quad i \neq j$$

We suppose now

$$v = p_1u_1 + p_2u_2 + p_3u_3 \quad and \quad \sum_1^3 p_i = 1 \tag{1}$$

where p_i the barycentric coordinates of the point M

$$v^2 = (\sum_1^3 p_iu_i)^2 \quad or \quad 1 = p_1^2 + p_2^2 + p_3^2 - p_1p_2 - p_2p_3 - p_3p_1 \tag{2}$$

From (1) follows that

$$1 = p_1^2 + p_2^2 + p_3^2 + 2p_1p_2 + 2p_2p_3 + 2p_3p_2 \tag{3}$$

we multiply the relation (2) by 2 and then we add it with (1). we take

$$p_1^2 + p_2^2 + p_3^2 = 1 \tag{4}$$

Also

$$v.u_1 = p_1 + (\frac{-1}{2})(p_2 + p_3) \quad or \quad v.u_1 = \frac{3p_1 - 1}{2}$$

then

$$v.p_1 u_1 = \frac{3p_1^2 - p_1}{2} \tag{5}$$

Adding with respect 1,2,3 and from (1),(4), (5) follows that

$$\sum_1^3 (u_i, v)^2 = 3/2 \tag{6}$$

(b)

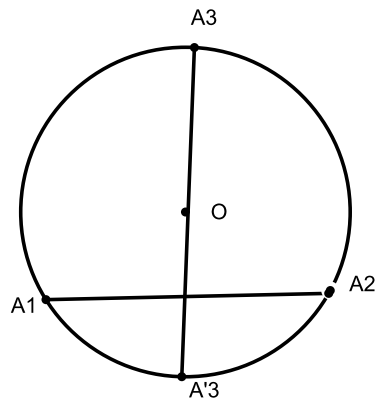
We now believe after the formula (6) the inequality of Cauchy- Schwarz will lead us directly to the solution. Let see it.

$$\frac{(u_1, v)^2 + (u_2, v)^2 + (u_3, v)^2}{3} \geq \left[\frac{|u_1.v| + |u_2.v| + |u_3.v|}{3}\right]^2$$

and then

$$\frac{3}{\sqrt{2}} \geq |u_1.v| + |u_2.v| + |u_3.v| \tag{7}$$

Is the problem solved?? But not. The Max of $|u_1.v| + |u_2.v| + |u_3.v|$ is when $|u_1.v| = |u_2.v| = |u_3.v|$. But this relation does not exist in Geometry.



We suppose that M is in the angle A_1OA_2 and $|u_1, v| = |u_2, v|$. Hence the point M must be the diametrical A'_1 of the point A_1, that is $A'_1 = M$.
But then $|u_3, v| = R = 1$, that is $|u_1, v| = |u_2, v| \neq |u_3, v|$. That is, the relation (7) is no true.
We will try otherwise. We see that $u_1.v + u_2.v + u_3.v = 0$, so we can suppose that $u_1v, u_2v \geq 0$, and $u_3.v \leq 0$. So we have:

$$\sum_1^3 |u_i.v| = (u_1 + u_2 - u_3)v = 2(u_1 + u_2)v \leq 2|u_1 + u_2||v| = 2|u_1 + u_2|$$

or

$$\left[\sum_1^3 |u_i.v|\right]^2 \leq 4|u_1 + u_2|^2 = 4(u_1^2 + u_2^2 + 2u_1.u_2) = 4(2 - 2\frac{1}{2}) = 4$$

Hence Max $\sum_1^3 |u_i.v| = 2$. For $M = A'_3$

The Generalization

The regular simplex $S_n = A_2A_2....A_{n+1}$ is inscribed in the unit sphere $(O, 1)$ in E^n. Let $OA_i = u_i$ and M on (O, R). Suppose that $\vec{OM} = v$. Find the

Max $\sum_{i=1}^{n+1} |u_i.v|$.
We work as in E^3 and we find that $u_i.u_j = -\frac{1}{n}$
For some index $m \in (1, 2, 3.....n)$ we will take

$$\sum_{i=1}^{n+1} |u_i.v| = \Big[\sum_{i=1}^{m} u_i - \sum_{i=m+1}^{n+1} u_i\Big].v = \Big[2\sum_{i=1}^{m} u_i\Big]v \leq 2|\sum_{i=1}^{m} u_i|$$

Also

$$|\sum_{i=1}^{m} u_i|^2 = m + 2\binom{m}{2}(-\frac{1}{n}) = \frac{m(n-m+1}{n}$$

The max for $m = [\frac{n+1}{2}]$ and equal to

$$\frac{n+1}{\sqrt{n}} \quad for \quad n = odd \qquad and \quad \sqrt{n+2} \quad for \quad n = even$$

P.S.
We can determine the point M_0 so that the vector $O\vec{M}_0$ gives the Max. We divide the points $A_1, A_2, ...A_{n+1}$ int two groups G_1, G_2 so that the first includes $[\frac{n+1}{2}]$ points and the other $n - [\frac{n+1}{2}]$ points. Let g_1, g_2 the cendtroids of G_1, G_2 respectively. The line $g_1 g_2$ intersects the circonscribed sphere to points M_0 and M_0'. The number $[\frac{n+1}{2}]$ determines and the number of solutions.

e.mail: gtsintsifas@yahoo.com
Blog: gtsintsifas