Integral point property of a convex body in the lattice plane.

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Joseph Hammer in (1) proved some analogous theorem to the well known theorem of Blichfeldt in Geometry of Numbers.
Blichfeldt 's theorem states that if the area of a plane set is greater than the one, then, the plane set can be translated so that would cover at least two lattice points.
Joseph Hammer substituted Blichfeldt 's translation with a rotation around special points. Especially he proved that if a convex set has area greater than $\frac{9}{8}\pi$ and its centroid concides with a lattice point, then by rotation around that point, the convex set can be brought into such a position that it covers beside the centroid, two further lattice points.
We will prove a similar theorm.
Let H be a convex set in a plane. For an interior point $M \in H$ there is a line , through the point M, so that the ratio

$$f(M) = \frac{AM}{AB} = min$$

where A, B the intersections of the line with the boundary of H. We call $k(H)$ the max of these min ratios, so that

$$max f(M) = k$$

The radio k is called "centralness coefficient" for H and such a position O of M "center" of H. That is

$$max f(M) = f(O) = k$$

B.H.Newmann () proved for a plane convex set that there is a unique "center" and that the "centralness coefficient" satisfies the inequality

$$\frac{1}{2} \leq k \leq \frac{2}{3}$$

see also (3),(4).
We will prove:

Theorem

Let H be a convex set in the plane so that

$$areaH \geq \frac{\pi}{4k^2}$$

where k the centralness coeficient and O its "center". We suppose that O consides with a lattice point. The convex set H is possible to be brought into such a position by rotation that it covers, besides O, two further lattice points lying in a str. line through the point O.

Proof

It is well known that for a convex set H holds:

$$|H| = areaH \leq \frac{d^2}{4}\pi \tag{1}$$

where d the diameter of H, see (2), (5).
Let A, B points of H , so that $(AB) = D$ and $AO \cap boundaryH = M$ and $BO \cap boundaryH = N$. The parallel line through the point O interscts AN, BM at the points K, L respectively.

From the similar triangles OML, AMB we will have:

$$\frac{OL}{AB} = \frac{OM}{AM} \geq \ k$$

Therefore using the relation (1) and our supposition we have:

$$OL \geq kd \geq 2k\sqrt{\frac{|H|}{\pi}} \geq 1$$

Similarly $OK \geq 1$. Hence it is well unterstood that by rotating H around O we will find such a position.
Comment
The generalization in E^n is easy. The centralness coeficent is $\frac{1}{2} \leq k \leq \frac{n}{n+1}$ see (1),if the volume of the convex set H is $V(H) \geq (2k)^n.k_n$ where k_n is the volume of the unit sphere in E^n, taking in our mint that $V(H) \leq k_n(\frac{d}{2})^n$ the solution is the same as in E^2

References

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