The inequality (condition) of Hadwiger.

G. A. Tsintsifas

Let F_0 and F_1 convex sets. We have defined "cutting number of F_0 with respect (or relative) to F_1" every real number r so that the rF_1 does not included in the interior of F_0 and does not includ in the interior the F_0. Let now an axis OX ant O_1, A_1 points of F_1.We suppose that $\angle(OX, O_1A_1) = \phi$.We understood that the cutting number r depents from the angle ϕ. So we can denote by $d(\phi)$ the widht of the cutting numbers (see the paper about the cutting numbers in this blog). The Wadwiger condition is about the relation between the perimeters and the areas of two convex sets so that exist the possibility of a transposition of the one in the interior of the other. (By transposition we mean a displacement or an isometry, that is a transformation product of translations, rotations and symmetries).
The continuity of the angle ϕ assures as that the continue fonction $d(\phi)$ has extrimities. We set

$$d_0 = d(\phi_0) = min.d(\phi)$$

Let now $\rho_0 = \rho_{F_1}(F_0)$ the inradius 0f F_0 relative to F_1 and $R_0 = R_{F_1}(F_0)$ for the angle ϕ_0. That is

$$d_0 = [\rho_0, R_0]$$

defination
We will define by Strong cutting number every real number of the of the interval $d_0 = [\rho_0, R_o]$.We denote r_0.
That is:

$$\rho_0 \leq r_0 \leq R_0$$

Hence, according the above, there is no transposition so that the r_0F_1 includes or is including to F_0. Let us suppose that $r_0 = 1$ that is the strong cutting number o F_0 relative to F_1 is the unity and that F_1 has an angle equal to θ with respect to constand axis. We know that the mixed area $V(F_0, F_1)$ is a function of θ. The integration of the relation

$$2V(F_0, F_1) \geq V(F_0) + V(F_1)$$

with respect θ from 0 to 2π, gives:

$$2.\int_0^{2\pi} V(F_0,F_1)d\theta \geq \int_0^{2\pi}\Big[V(F_0)+V(F_1)\Big]$$

or

$$2\frac{L_0L_1}{2} \geq \ 2\pi\Big[V(F_0)+V(F_1)\Big]$$

Therefore

$$L_0.L_1 \geq 2\pi\Big[V(F_0)+V(F_1)\Big] \tag{1}$$

hence, if for F_0 and F_1 is $r_0 = 1$ (that is strong cutting number=1) then we will have the relation (1). Of course ,it is not correct the opposite.
The inequality (1) assures us that if

$$L_0L_1 < 2\pi\Big[V(F_0)+V(F_1)\Big] \tag{2}$$

the the unit is not a strong number. that is will exist a transposition so that the F_1 will be in the interior of F_0 or the opposit.
The relation (2) is called inequality or condition o Hadwinger.

Some applications

1. Let F_o, F_1 convex sets in E^2 and r_0 strong cutting number of F_0 to F_1.The relation (1) is

$$r_0L_0L_1 \geq 2\pi\Big[V(F_0)+r_0^2V(F_1)\Big] \tag{3}$$

or

$$0 \geq 2\pi.V(F_1).r_0^2 - L_0.L_1.r_0 + 2\pi.V(F_0)$$

We now denote by ρ_0 the strong inradius of F_0 with respect F_1 and by R_0 the strong circumradius of F_0 with respect of F_1. From (3) we easily we take.

$$\frac{L_0^2.L_1^2}{16\pi^2} - V(F_0).V(F_1) \geq \frac{V(F_1)^2.\Big[R_0-\rho_0\Big]^2}{4} \tag{4}$$

Let $t \notin d_0 = [\rho_0, R_0]$ For the sets F_0, tF_1 the Hadwiger's condition is:

$$0 < 2\pi V(F_1).t^2 - L_0.L_1.t + 2\pi V(F_0)$$

the number t must be beetwin the roots of the equation

$$0 = 2\pi V(F_1).t^2 - L_0.L_1.t + 2\pi V(F_0)$$

that is:

$$t < \frac{L_0 L_1 - \sqrt{D}}{4\pi V(F_1)} \qquad and \qquad t > \frac{L_0 L_1 + \sqrt{D}}{4\pi V(F_1)}$$

where $D = L_0^2 L_1^2 - 16\pi^2 V(F_0)(F_1)$.
Hence, for

$$t = \frac{L_0 L_1 - \sqrt{D}}{4\pi V(F_1)} - \varepsilon$$

and for ε small the figure tF_1 can insert in F_0. For $\varepsilon \to 0$, we finally see that there is

$$t \geq \frac{L_0 L_1 - \sqrt{D}}{4\pi V(F_1)}$$

so that the tF_1 could be inside to F_0.
Exactly the same for

$$t \leq \frac{L_0 L_1 + \sqrt{D}}{4\pi V(F_1)}$$

we can displace F_1 so that F_1 inclube the F_9 **2**. For the convex figures F_0, F_1 in the plane holds

$$L_0^2 L_1^2 \geq 2\pi \Big[L_1^2 V(F_0) + L_0^2 V(F_1) \Big] \tag{5}$$

The proof is an easy application of the formula (1) for the F_0 and $F = \frac{L_0}{L_1}.L_1$. Indeed the convex figures F_0 and F have the same perimeters, therefore we can use (1). that is

$$L_0 L \geq 2\pi \Big[V(F_0) + V(F) \Big]$$

or

$$L_0 \Big(\frac{L_0}{L_1} L_1 \Big) \geq 2\pi \Big[V(F_0) + V(F_1). \frac{L_0^2}{L_1^2} \Big]$$

In the paper "The cutting Numbers " in this blog the relation (6) is

$$rV(F_0, F_1) \geq \frac{V(F_0) + r^2 V(F_1)}{2}$$

where $r = r_{F_1}(F_0)$.
We suppose that the F_0, F have the same breadth in the direction θ, and F_0, F_1 have Breadth $B_0(\theta_1), B_1(\theta_1)$. The figure $F = \frac{B_0(\theta_1)}{B_1(\theta_1)} F_1$ has the same

breadth with F_0 relative the direction θ_1 therefore the above for $r = \frac{B_0(\theta_1)}{B_1(\theta_1)}$ is.

$$V(F_0, F_1)^2 - V(F_0)V(F_1) \geq \frac{\Big[B_1(\theta_1)^2 V(F_0) - B_0(\theta_1)^2 V(F_1)\Big]^2}{4B_0(\theta_1)^2 B_1(\theta_1)^2} \qquad (6)$$

Some interest has the application of the above for $F_1 = U$ the unit circle.

References.

1. Cutting Numbers, G.Tsintsifas, this Blog.
2. Introduction to Integral Geometry, L. A. Santalo, Hermann Editeurs.