Geometry Of Simplexes

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We present in this paper some propositions for a n-dimensional simplex.

We use the following symbols.

The points $A_1, A_2, A_3, \ldots, A_{n+1}$ are the vertexes of the n-dimensional simplex $\sigma^{(n)}$ in $R^{(n)}$.

$V^{(n)}$: The n-dimensional linear mesure of $\sigma^{(n)}$. (The volume in Euclidean metric space).

$\sigma_i^{(n-1)}$: The (n-1)-dimensional simplex (or the (q-1)-face) opposite to the vertex A_i.

$V_i^{(n-1)}$: The volume of the $\sigma_i^{(n-1)}$.

$a_{ij} = a_{ji}$: The edge $A_i A_j$.

If $p \in \sigma^{(n)}$ $V_i^{(n)}(p)$ is the volume of the simplex with vertexes $A_1, A_2, \ldots, A_{i-1}, p, A_{i+1}, \ldots, A_{n+1}$.

If $p_i \in \sigma_i^{(n-1)}$ $V_j^{(n-1)}(p_i)$ is the volume of the simplex $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{j-1}, p_i, A_{j+1}, \ldots, A_{n+1}$.

h_i: The distance of the vertex A_i from the $\sigma_i^{(n-1)}$, and d_i the distance of the point $p \in \sigma^{(n)}$ from $\sigma_i^{(n-1)}$.

$A_i p = x_i$, $p p_i = u_i$.

Proposition 1.

Let $6^{(n)}$ be a simplex in $R^{(n)}$ with vertexes A_1, $A_2, \ldots, A_{n+1}$. If p is an interior point and $p_i = A_i p \cap 6_i^{(n-1)}$, the product

$$V_{j_1}^{(n-1)}(p_1)\, V_{j_2}^{(n-1)}(p_2) \ldots V_{j_{n+1}}^{(n-1)}(p_{n+1})$$

is constant for $j_i \in [1,2,3,\ldots,n+1] \wedge j_e \neq j_k \wedge j_\lambda \neq \lambda$.

PROOF :

We use the barycentric coordinates of the affine geometry.

If $p \in 6^{(n)}$ (interior point) we have

$$p = \sum_{i=1}^{n+1} \lambda_i A_i \tag{1}$$

where $\lambda_i > 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$.

From (1) we obtain

$$p = \lambda_i A_i + (1-\lambda_i) \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \frac{\lambda_j A_j}{1-\lambda_i} \tag{2}$$

and consequently

$$p_i = \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \frac{\lambda_j A_j}{1-\lambda_i} \tag{3}$$

2. Lemma.

If p is an interior point in a $6^{(m)}$-simplex $A_1, A_2, \ldots, A_{m+1}$ and $p = \sum_{i=1}^{m+1} q_i A_i$ $\left(q_i > 0,\ \sum_{i=1}^{m} q_i = 1\right)$ the ratio $\frac{V_i^m(p)}{q_i}$ has the same arithmetical value for $i \in [1,2,\ldots,m+1]$.

In to prove this we use the method of the mathematical induction.

For $m=2$ the proof is elementary.

We assume that the proposition is true for $m=k$ and let $p_i = A_i p \cap 6_i$.

We have $p_i = \sum_{\substack{j=1 \\ j\neq i}}^{k+1} \frac{q_j}{1-q_i} A_j$ and from the assumption for the $6_i^{(k-1)}$-simplex we have

$$\frac{V_1^{(\kappa-1)}(p_i)}{q_1} = \frac{V_2^{(\kappa-1)}(p_i)}{q_2} = \dots = \frac{V_{i-1}^{(\kappa-1)}(p_i)}{q_{i-1}} = \frac{V_{i+1}^{(\kappa-1)}(p_i)}{q_{i+1}} = \dots = \frac{V_{\kappa+1}^{(\kappa-1)}(p_i)}{q_{\kappa+1}} \tag{4}$$

We multiply successively the relations (4) with $\frac{h_i}{\kappa}$ and $\frac{d_i}{\kappa}$ and subtract the corresponding terms. Taking into account the relation

$$\frac{1}{\kappa} h_i V_e^{(\kappa-1)}(p_i) - \frac{1}{\kappa} d_i V_e^{(\kappa-1)}(p_i) = V_e^{\kappa}(p) \quad (e \neq i)$$

we have

$$\frac{V_1^{(\kappa)}(p)}{q_1} = \frac{V_2^{(\kappa)}(p)}{q_2} = \dots = \frac{V_{i-1}^{(\kappa)}(p)}{q_{i-1}} = \frac{V_{i+1}^{(\kappa)}(p)}{q_{i+1}} = \dots = \frac{V_{\kappa+1}^{(\kappa)}(p)}{q_{\kappa+1}} .$$

We work now in the same way with the simplex $6_\lambda^{(\kappa-1)}$ $(\lambda \neq i)$ and we obtain the desired result.

3. We return now to the proof of the theorem. Using the above lemma we have for the simplexes $6_1^{(n-1)}, 6_2^{(n-1)}, \dots, 6_{n+1}^{(n-1)}$ the relations

$$\begin{aligned}
&\frac{V_2^{(n-1)}(p_1)}{\lambda_2} = \frac{V_3^{(n-1)}(p_1)}{\lambda_3} = \frac{V_4^{(n-1)}(p_1)}{\lambda_4} = \dots = \frac{V_{n+1}^{(n-1)}(p_1)}{\lambda_{n+1}} \\
&\frac{V_1^{(n-1)}(p_2)}{\lambda_1} = \frac{V_3^{(n-1)}(p_2)}{\lambda_3} = \dots\dots\dots = \frac{V_{n+1}^{(n-1)}(p_2)}{\lambda_{n+1}} \\
&\frac{V_1^{(n-1)}(p_3)}{\lambda_1} = \dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots \\
&\frac{V_1^{(n-1)}(p_{n+1})}{\lambda_1} = \frac{V_2^{(n-1)}(p_{n+1})}{\lambda_2} = \dots\dots\dots = \frac{V_n^{(n-1)}(p_{n+1})}{\lambda_n}
\end{aligned} \tag{5}$$

We change the order in each line of (5) in such a way that the indices of the λ's appearing in the denominator to be all different in each column. If now we multiply the terms in each column we obtain.

$$V_{j_1}^{(n-1)}(p_1)\, V_{j_2}^{(n-1)}(p_2) \dots V_{j_{n+1}}^{(n-1)}(p_{n+1}) \quad \text{const.}$$

Proposition 2.

Let p be a point in the interior of the simplex $\sigma^{(n)}$ in $R^{(n)}$ and $p_i = A_i p \cap \sigma_i^{(n-1)}$

We call $V^{(n)}(p)$ the volume of the n-simplex with vertexes $p_1, p_2, \ldots, p_{n+1}$. We will prove that

$$\left|V^{(n)}(p)\right| \leqslant \frac{1}{n^n}\left|V^{(n)}\right|.$$

PROOF.

If $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are the barycentric coordinates of the point p we have

$$p = \sum_{i=1}^{n+1} \lambda_i A_i \tag{1}$$

and $\lambda_i > 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$.

We know (proposition 1.3)

$$p_i = \sum_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\lambda_j A_j}{1-\lambda_i}. \tag{2}$$

Let $x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in}$ and $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$ be the coordinates of the points p_i and A_i in an orthogonal system.

We have from (2)

$$x_{kj} = \sum_{\substack{i=1\\ i\neq k}}^{n+1} \frac{\lambda_i}{1-\lambda_k}\, a_{ij} \tag{3}$$

$k = [1, 2, \ldots, n+1]$ and $j = [1, 2, \ldots, n]$.

We know that

$$V^{(n)}(p) = \frac{1}{n!}\begin{vmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1n} \\ 1 & x_{21} & x_{22} & \cdots & x_{2n} \\ 1 & x_{31} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_{n+1,1} & \cdots & \cdots & x_{n+1,n} \end{vmatrix} \tag{4}$$

The substitution of (3) in (4) will give

$$\prod_{i=1}^{n+1}(1-\lambda_i)V^{(n)}(p) = \begin{vmatrix} 0 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n+1} \\ \lambda_1 & 0 & \lambda_3 & \cdots & \lambda_{n+1} \\ \lambda_1 & \lambda_2 & 0 & \cdots & \lambda_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & 0 \end{vmatrix} V^{(n)}$$

and $\prod_{i=1}^{n+1}(1-\lambda_i)V^{(n)}(p) = (-1)^n n \prod_{i=1}^{n+1}\lambda_i V^{(n)}$.

The relation between the arithmetical and geometrical mean is

$$1-\lambda_i = \sum_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j \geqslant n \prod_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j^{\frac{1}{n}}$$

Therefore

$$|V^{(n)}(p)| \leqslant \left| \frac{(-1)^n n \prod_{i=1}^{n+1}\lambda_i}{n^{n+1}\prod_{i=1}^{n+1}\lambda_i} \right| |V^{(n)}|$$

and

$$|V^{(n)}(p)| \leqslant \frac{1}{n^n}|V^{(n)}|.$$

Equality holds if and only if p is the barycenter.

Proposition 3.

Let p be a point in the simplex $\sigma^{(n)}$ We put $(A_ip) = x_i$, $(pp_i) = u_i$ with $p_i = A_ip \cap \sigma_i^{(n-1)}$ We will prove

1. $\sum_{i=1}^{n+1} \frac{x_i}{u_i} \geqslant n(n+1)$
2. $\prod_{i=1}^{n+1} x_i \geqslant n^n \prod_{i=1}^{n+1} u_i$

3. $\sum_{i=1}^{n+1} x_i \geqslant 2 \sum_{j>\lambda}^{1,n+1} u_j^{\frac{1}{2}} u_\lambda^{\frac{1}{2}}$

4. $\sum_{i=1}^{n+1} x_i^2 \geqslant 2 \sum_{i>j}^{1,n+1} u_i u_j + n(n^2-1) \prod_{i=1}^{n+1} u_i^{\frac{n+1}{2}}$

5. $\sum_{i>j}^{1,n+1} \frac{x_i x_j}{u_i u_j} \geqslant \frac{n^3(n+1)}{2}$

6. $\sum_{i>j}^{1,n+1} x_i x_j \geqslant \sum_{i>j}^{1,n+1} u_i u_j + \frac{1}{2} n(n^2-1)(n+1) \prod_{i=1}^{n+1} u_i^{\frac{n+1}{2}}$

7. $(n-1) \sum_{i=1}^{n+1} x_i + n \sum_{i=1}^{n+1} u_i \geqslant 2 \sum_{i>j}^{1,n+1} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}}$

PROOF.

Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n+1}$ be the barycentric coordinates of the point p. We have

$$p = \sum_{i=1}^{n+1} \lambda_i A_i \quad \text{and} \quad \lambda_i > 0, \ \sum_{i=1}^{n+1} \lambda_i = 1.$$

1. Since $p = \lambda_i A_i + (1-\lambda_i) \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \frac{\lambda_j A_j}{1-\lambda_i} = \lambda_i A_i + (1-\lambda_i) p_i$, we have that $\frac{x_i}{u_i} = \frac{1-\lambda_i}{\lambda_i} = \frac{1}{\lambda_i} \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j$.

Hence

$$\sum_{i=1}^{n+1} \frac{x_i}{u_i} = \sum_{i>j}^{1,n+1} \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} \right).$$

Taking into account that $\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} \geqslant 2$ we obtain

$$\sum_{i=1}^{n+1} \frac{x_i}{u_i} \geqslant \frac{n(n+1)}{2} \cdot 2 = n(n+1)$$

2. $\frac{\prod_{i=1}^{n+1} x_i}{\prod_{i=1}^{n+1} u_i} = \frac{\prod_{i=1}^{n+1} (1-\lambda_i)}{\prod_{i=1}^{n+1} \lambda_i}$, but $1-\lambda_i = \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j \geqslant n \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j^{\frac{1}{n}}$ and $\prod_{i=1}^{n+1} (1-\lambda_i) \geqslant n^n \prod_{i=1}^{n+1} \lambda_i$ etc ...

3. $\sum_{i=1}^{n+1} x_i = \sum_{j>q}^{1,n+1} \left(\frac{\lambda_j}{\lambda_q} u_q + \frac{\lambda_q}{\lambda_j} u_j \right)$, but $\frac{\lambda_j}{\lambda_q} u_q + \frac{\lambda_q}{\lambda_j} u_j \geqslant 2 u_q^{\frac{1}{2}} u_j^{\frac{1}{2}}$;

therefore

$$\sum_{i=1}^{n+1} x_i \geqslant 2 \sum_{j>\lambda}^{1,n+1} u_\lambda^{\frac{1}{2}} u_j^{\frac{1}{2}}.$$

4. $$\sum_{i=1}^{n+1} x_i^2 = \sum_{i>j}^{1,n+1} \left(\frac{\lambda_i^2}{\lambda_j^2} u_j^2 + \frac{\lambda_j^2}{\lambda_i^2} u_i^2 \right) + 2 \sum_{\substack{i>j \\ i,j\neq q}}^{1,n+1} \frac{\lambda_i \lambda_j}{\lambda_q^2} u_q^2 \quad , \text{ but}$$

$$\frac{\lambda_i^2}{\lambda_j^2} u_j^2 + \frac{\lambda_j^2}{\lambda_i^2} u_i^2 \geqslant 2 u_i u_j \quad \text{and} \quad 2 \sum_{\substack{i>j \\ i,j\neq q}}^{1,n+1} \frac{\lambda_i \lambda_j}{\lambda_q^2} u_q^2 \geqslant n(n^2-1) \prod_{i=1}^{n+1} u_i^{\frac{n+1}{2}}.$$

5. $$\frac{x_i}{u_i} = \frac{1}{\lambda_i} \sum_{\substack{k=1 \\ k\neq i}}^{n+1} \lambda_k \geqslant \frac{n \prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n}}}{\lambda_i^{\frac{n+1}{n}}} . \quad \text{Therefore} \quad \frac{x_i x_j}{u_i u_j} \geqslant n^2 \frac{\prod_{k=1}^{n+1} \lambda_k^{\frac{2}{n}}}{\lambda_i^{\frac{n+1}{n}} \lambda_j^{\frac{n+1}{n}}} .$$

If we add all the terms for $i,j = 1,2,\dots,n+1$ for $i>j$ we obtain

$$\sum_{i>j}^{1,n+1} \frac{x_i x_j}{u_i u_j} \geqslant n^2 \left[\prod_{k=1}^{n+1} \lambda_k^{\frac{2}{n}} \right] \left[\sum_{i>j}^{1,n+1} \lambda_i^{-\frac{n+1}{n}} \lambda_j^{-\frac{n+1}{n}} \right],$$

but

$$\sum_{i>j}^{1,n+1} \lambda_i^{-\frac{n+1}{n}} \lambda_j^{-\frac{n+1}{n}} \geqslant \frac{n(n+1)}{2} \prod_{k=1}^{n+1} \lambda_k^{-\frac{2}{n}} .$$

Therefore

$$\sum_{i>j}^{1,n+1} \frac{x_i x_j}{u_i u_j} \geqslant \frac{n^3(n+1)}{2} \prod_{k=1}^{n+1} \lambda_k^{\frac{2}{n}} \lambda_k^{-\frac{2}{n}} = \frac{n^3(n+1)}{2}$$

6. $$x_i x_j = \left[\left(\sum_{\substack{k=1 \\ k\neq i,j}}^{n+1} \lambda_k \right) : (\lambda_i \lambda_j) \right] u_i u_j + u_i u_j \quad \text{because} \quad \sum_{i=1}^{n+1} \lambda_i = 1 .$$

Consequently $$x_i x_j \geqslant (n-1) \frac{\prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n-1}}}{\lambda_i^{\frac{n}{n-1}} \lambda_i^{\frac{n}{n-1}}} u_i u_j + u_i u_j \qquad (1)$$

Using the relation

$$\sum_{i>j}^{1,n+1} \frac{u_i u_j}{\lambda_i^{\frac{n}{n-1}} \lambda_j^{\frac{n}{n-1}}} \geqslant \frac{n(n+1)}{2} \frac{\prod_{k=1}^{n+1} u_k^{\frac{2}{n+1}}}{\prod_{k=1}^{n+1} \lambda_k^{-\frac{2n}{(n+1)(n-1)}}} \qquad (2)$$

we obtain from (1)

$$\sum_{i>j}^{1,n+1} x_i x_j \geqslant \sum_{i>j}^{1,n+1} u_i u_j + \frac{n(n^2-1)}{2} \prod_{k=1}^{n+1} \lambda_k^{-\frac{1}{n+1}} u_k^{\frac{2}{n+1}} , \tag{3}$$

But

$$1 = \sum_{i=1}^{n+1} \lambda_i \geqslant (n+1) \prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n+1}}$$

and the relation (3) will give

$$\sum_{i>j}^{1,n+1} x_i x_j \geqslant \sum_{i>j}^{1,n+1} u_i u_j + \frac{n(n+1)(n^2-1)}{2} \prod_{k=1}^{n+1} u_k^{\frac{2}{n+1}}$$

7. We have $u_i = \frac{\lambda_i}{1-\lambda_i} x_i$. Therefore

$$(n-1)\sum_{i=1}^{n+1} x_i + n \sum_{i=1}^{n+1} \frac{\lambda_i}{1-\lambda_i} x_i = \sum_{i=1}^{n+1} \frac{(n-1)(1-\lambda_i)+n\lambda_i}{1-\lambda_i} x_i = \sum_{i=1}^{n+1} \frac{n-(1-\lambda_i)}{1-\lambda_i} x_i =$$

$$= n \sum_{i=1}^{n+1} \frac{x_i}{1-\lambda_i} - \sum_{i=1}^{n+1} x_i ,$$

But $n = \sum_{i=1}^{n+1} (1-\lambda_i)$ and we obtain (Schwarz)

$$n \sum_{i=1}^{n+1} \frac{x_i}{1-\lambda_i} \geqslant \left[\sum_{i=1}^{n+1} x_i^{\frac{1}{2}} \right]^2 = \sum_{k=1}^{n+1} x_k + 2 \sum_{i>j}^{1,n+1} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}} ,$$

or

$$(n-1)\sum_{i=1}^{n+1} \lambda_i + n \sum_{i=1}^{n+1} u_i \geqslant 2 \sum_{i>j}^{1,n+1} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}} .$$

Proposition 4.

Let $\sigma^{(n)}$ be a n-simplex in $R^{(n)}$ and p a point in the interior of $\sigma^{(n)}$. Let the midpoints of the segments $A_1p, A_2p, \ldots, A_{n+1}p$ be $\mu_1, \mu_2, \ldots, \mu_{n+1}$. We consider the hyperplane π_i which pass through μ_i and is parallel to hyperplane in which lies the $\sigma_i^{(n-1)}$. This cuts from $\sigma^{(n)}$ a n-simplex which has A_i as one of its vertexes. If $V_{\mu_i}^{(n)}$ is the volume of this simplex, we shall prove

$$\sum_{i=1}^{n+1} V_{\mu_i}^{(n)} \geqslant \left(\frac{n}{2}\right)^2 \frac{V^{(n)}}{(n+1)^{n-1}}.$$

PROOF.

We set $p=\sum_{i=1}^{n+1}\lambda_i A_i$ with $\lambda_i>0$ and $\sum_{i=1}^{n+1}\lambda_i=1$. We have

$$\mu_i=\frac{1}{2}(A_i+p)=\frac{1}{2}\left[\lambda_1A_1+\lambda_2A_2+\cdots+\lambda_{i-1}A_{i-1}+(1+\lambda_i)A_i+\lambda_{i+1}A_{i+1}+\cdots+\lambda_{n+1}A_{n+1}\right]$$

and

$$\frac{(A_i\mu_i)}{(A_ip_i)}=\frac{1-\lambda_i}{2}\qquad (p_i=A_ip\cap\sigma_i^{(n-1)}).$$

Therefore

$$V_{\mu_i}^{(n)}=\left[\frac{1-\lambda_i}{2}\right]^n V^{(n)}.$$

We sum with respect to i from 1 to $n+1$ and we obtain

$$\sum_{i=1}^{n+1}V_{\mu_i}^{(n)}=\frac{1}{2^n}V^{(n)}\sum_{i=1}^{n+1}(1-\lambda_i)^n$$

Taking into account the relation

$$\sum_{i=1}^{n+1}(1-\lambda_i)^n\geqslant\frac{\left[\sum_{i=1}^{n+1}(1-\lambda_i)\right]^n}{(n+1)^{n-1}}=\frac{n^n}{(n+1)^{n-1}}$$

we obtain the desired result.

Equality holds if and only if p is the barycenter.

Proposition 5.

Let $\sigma^{(n)}$ be a n-simplex in $R^{(n)}$ with vertexes $A_1, A_2, \ldots, A_{n+1}$ and $\sigma^{(q)}$ a q-face with vertexes $A_1, A_2, \ldots, A_{q+1}$ $(q<n)$. If $p=\sum_{i=1}^{n+1}\lambda_iA_i$, $\lambda_i\geqslant 0$ $\sum_{i=1}^{n+1}\lambda_i=1$, and

$q=\sum_{j=1}^{q+1}\mu_j A_j$ we shall prove that

$$|p-q| \leqslant \sum_{i=1}^{n+1}\sum_{j=1}^{q+1}\lambda_i\mu_j d$$

with $i \neq j$, where d is the diameter of $\sigma^{(n)}$.

PROOF.

We have $q-p=\sum_{j=1}^{q+1}\mu_j A_j - p = \sum_{j=1}^{q+1}\mu_j A_j - \sum_{j=1}^{q+1}\mu_j p = \sum_{j=1}^{q+1}\mu_j (A_j - p)$. (1)

using

$$A_j - p = \sum_{i=1}^{n+1}\lambda_i A_j - \sum_{i=1}^{n+1}\lambda_i A_i = \sum_{i=1}^{n+1}\lambda_i (A_j - A_i)$$

we obtain from (1)

$$q-p=\sum_{j=1}^{q+1}\sum_{i=1}^{n+1}\lambda_i\mu_j (A_j - A_i) \tag{2}$$

The number of the terms in the right-hand side of (2) is $(q+1)(n+1)$. Of this summand the $q+1$, are equal to zero, and $|A_j - A_i| \leqslant d$. So we have

$$|q-p| \leqslant \sum_{i=1}^{n+1}\sum_{j=1}^{q+1}\lambda_i\mu_j d$$

for $i \neq j$. If we take for p, q the centroids, then the above relation reduces to $|q-p| \leqslant \frac{n}{n+1} d$.

(ALEKSANDROV : Combinatorial Topology page 214)

The Leibnitz formula and $\sigma^{(n)}$- simplex.

Let $A_1, A_2, \ldots, A_n$ be n points in $R^{(d)}$ with masses $m_1, m_2, \ldots, m_n$ and let G be the barycenter. (That is for a point O $\overrightarrow{OG} = \frac{m_1\overrightarrow{OA_1}+m_2\overrightarrow{OA_2}+\cdots+m_n\overrightarrow{OA_n}}{m_1+m_2+\cdots+m_n}$). If p is any point in $R^{(d)}$, the following relation holds

$$\sum_{i=1}^{n} m_i \overrightarrow{pA_i}^2 = m\overrightarrow{pG}^2 + \sum_{i=1}^{n} m_i \overrightarrow{GA_i}^2 \text{ (Leibniz formula)} \tag{1}$$

$$m = m_1 + m_2 + \cdots + m_n.$$

We set $(A_i A_j) = (A_j A_i) = a_{ij} = a_{ji}$.

We shall prove the following formula

$$\sum_{i=1}^{n} m_i \overrightarrow{PA_i}^2 = m\,\overrightarrow{PG}^2 + \frac{1}{m} \sum_{i>j}^{1,n} m_i m_j a_{ij}^2 . \qquad (2)$$

It suffices to prove that

$$\sum_{i=1}^{n} m_i \overrightarrow{GA_i}^2 = \frac{1}{m} \sum_{i>j}^{1,n} m_i m_j a_{ij}^2 \qquad (3)$$

We use the mathematical induction.

The relation (3) is elementary for $n=2$. So for any two points $A_1(m_1)$ and $A_2(m_2)$ we have

$$m_1 \overrightarrow{GA_1}^2 + m_2 \overrightarrow{GA_2}^2 = \frac{m_1 m_2 a_{12}}{m_1+m_2} \qquad (4)$$

We suppose that the proposition holds for $n=k$.

Let G be the barycenter of the points $A_1(m_1)$, $A_2(m_2), \ldots, A_K(m_K)$ and G' the barycenter of the points $A_1(m_1), A_2(m_2), \ldots, A_K(m_K), A_{K+1}(m_{K+1})$.

For the two points $G\,(m = m_1+m_2+\cdots+m_K)$ and $A_{K+1}(m_{K+1})$ we have, according to (4)

$$m\overrightarrow{G'G}^2 + m_{K+1} \overrightarrow{G'A_{K+1}}^2 = \frac{m\,m_{K+1}\overrightarrow{GA}_{K+1}^2}{m+m_{K+1}} \qquad (5)$$

We put in (2) $p = G'$ and obtain

$$\sum_{i=1}^{K} m_i \overrightarrow{G'A_i}^2 = m\overrightarrow{G'G}^2 + \frac{1}{m} \sum_{i>j}^{1,K} m_i m_j a_{ij}^2 \qquad (6)$$

We add (5) and (6)

$$\sum_{i=1}^{K+1} m_i \overrightarrow{G'A_i}^2 = \frac{m m_K \overrightarrow{GA_{K+1}}^2}{m+m_{K+1}} + \frac{1}{m} \sum_{i>j}^{1,K} m_i m_j a_{ij}^2 \qquad (7)$$

We put in relation (2) $p = A_{K+1}$ and we solve for $\overrightarrow{GA}_{K+1}^2$ and we substitute it in (7). After the simplifications we have

$$\sum_{i=1}^{n+1} m_i \overrightarrow{G'A_i}^2 = \frac{1}{m+m_{k+1}} \sum_{i>j}^{1,k+1} m_i m_j a_{ij}^2$$

We now apply the formulas (1) and (2) to a $6^{(n)}$-simplex.

We consider the $6^{(n)}$-simplex in $R^{(n)}$ with vertexes $A_1, A_2, \ldots, A_{n+1}$ and let p be an interior point.

We have $p = \sum_{i=1}^{n+1} \lambda_i A_i$ and $\lambda_i > 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$.

The formula (1) for a point q becomes

$$\sum_{i=1}^{n+1} \lambda_i \overrightarrow{qA_i}^2 = \overrightarrow{pq}^2 + \sum_{i=1}^{n+1} \lambda_i \overrightarrow{pA_i}^2 \tag{8}$$

or

$$\sum_{i=1}^{n+1} \lambda_i \overrightarrow{qA_i}^2 \geqslant \sum_{i=1}^{n+1} \lambda_i \overrightarrow{pA_i}^2 \tag{9}$$

Equality for $q = p$.

If q coincides with the center of the n-circumsphere and $(qA_i) = R$ we have

$$R^2 \geqslant \sum_{i=1}^{n+1} \lambda_i \overrightarrow{pA_i}^2 \tag{10}$$

If we use the formula (2), we obtain

$$\sum_{i=1}^{n+1} \lambda_i \overrightarrow{qA_i}^2 = \overrightarrow{pq}^2 + \sum_{i>j}^{1,n+1} \lambda_i \lambda_j a_{ij}^2 \tag{11}$$

$$\sum_{i=1}^{n+1} \lambda_i \overrightarrow{qA_i}^2 \geqslant \sum_{i>j}^{1,n+1} \lambda_i \lambda_j a_{ij}^2 \tag{12}$$

$$R^2 \geqslant \sum_{i>j}^{1,n+1} \lambda_i \lambda_j a_{ij}^2 \tag{13}$$

A large number of inequalites can be obtained from the relations 9, 10, 11, 12 and 13 e.g.

a) $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \lambda_j a_{ij}^2 > (n+1)\sum_{i=1}^{n+1} \lambda_i \overrightarrow{pA_i}^2$

β) $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \lambda_i a_{ij}^2 > (n+1)\sum_{i>j}^{1,n+1} \lambda_i \lambda_j a_{ij}^2$

c) $R \geqslant \dfrac{\sum_{i>j}^{1,n+1} a_{ij}}{(n+1)\sqrt{n(n+1)}}$

d) $R \geqslant \dfrac{1}{S}\left[\sum_{i>j}^{1,n+1} V_i^{(n-1)} V_j^{(n-1)} a_{ij}^2\right]^{\frac{1}{2}}$ with $S = \sum_{i=1}^{n+1} V_i^{(n-1)}$ etc.

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