Equal circles, spheres

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Problem 1.
Three equal circles $(w_1, 1), (w_2, 1), (w_3, 1)$ have a common point H and pass from the points A, B, C . Prove that the point H is the orthocenter of the triangle ABC and the cirle ABC has radius equal 1.
We consider an orthogonal Kartesian system with origin the point H.
We will denote the $vector HM$ $=M$
so we have
$A = w_2 + w_3$, $B = w_1 + w_3$, $C = w_1 + w_2$
that is because Aw_2Hw_3, Bw_1Hw_3 are rhombus etc.
So we have $B - C = w_1 + w_3 - (w_1 + w_2) = w_3 - w_2$,
that is $(w_2 + w_3)(w_2 - w_3) = 0$, or HA is orthogonal to BC
The same for HB orthogonal to CA and CH orthogonal to AB. Therefore we see that H is the orthocenter of the triangle ABC.
We will find the cirvumradius of the triangleABC.
Let G the barycenter and K the circumceter of ABC. We know from the Euler line that $\vec{HG}$=2$\vec{GK}$,
that is

$$2/3(w_1 + w_2 + w_3) = 2K - 4/3(w_1 + w_2 + w_3)$$

Finally the circumradius is $|K - A| = |w_1 + w_2 + w_3 - A| = 1$

Problem 2.
Can we see the problem 1 as a generaligation in three dimensions ?
We consider four equal spheres $(w_1), (w_2), (w_3), (w_4)$ through the point H and intersecting every three spheres at the points A,B,C,D. What can we say about the point H, the circumradius of the tetrahedron ABCD ?
(a). About the point H. We prove that H is not the orthocenter.

The spheres w_1, w_2, w_3, w_4 form a network with center H and radical axes HA,HB,HC,HD. The plane $w_2w_3w_4$ is orthogonal and bisects at the pointA_1 the str. line segment HA. We see thatA_1 is the circucenter of the triangle $w_2w_3w_4$.. Similarly we take the points B_1, C_1, D_1. The homothetic transformation with ceter H ant ratio 2/1 transforms$B_1C_!D_!$ to BCD. But the planes $B_1C_1D_1$ and $w_2w_3w_4$ are not parallel. Therefore HA is not orthogonal to the plane BCD.
For the circumradius R of the tetrahedron ABCD we have to remark: From the triangle Hw_2A, it is $2 > HA$ hence the sphere (H,2) includes ABCD. So we have R < 2.