The Davenport Hajos's Theorem

G. A. Tsintsifas

A nice problem of the combinatorial Geometry is the theorem of Davenport-Hajos. It is about $n+2$ points plus one in E^n. That is for the points $A_1, A_2, ...A_{n+2}$ in E^n and for every point O, one, at least from the angles A_iOAj is non obtuse. That is one at least is: $\angle A_iOA_j \leq \frac{\pi}{2}$.

Proof

We will use the theorem of Helly. In E^n the ball is $B^n = (x_1^2 + x_2^2 + ...x_n^2 \leq 1)$ and the sphere $S^{n-1} = (x_1^2 + x_2^2 + ...x_n^2 = 1)$.The theorem of Helly is:

In S^{n-1} there are $F_1, F_2, ...F_k, \quad k \geq n+2$ convex sets, so that every $n+1$ have a common point, then all the convex sets will have a common point. For the simplicity we will prove the Davenport Helly's theorem for $n = 3$. The proof of the general case is similar.

So, in E^3 we have five points A_1, A_2, A_3, A_4, A_5 and we will show that for every point O we have at least $A_iOA_j \leq \frac{\pi}{2}$ for some $i \neq j \in (1, 2, 3, 4, 5)$

We can suppose, without any loss of the generality, that the points $A_i, i = 1, 2, 3, 4, 5$ are in the surface of the unit ball, the sphere $S^2 = x_1^2 + x_2^2 + x_3^2 = 1$.

We will use the reductio ab absurdium method. We accept for a moment that $\angle A_iOA_j > \frac{\pi}{2}, i \neq j$

We take A_1. We have: $arc(A_1A_j) > \frac{\pi}{2}$. Therefore the points A_2, A_3, A_4, A_5, will be in the opposit semisphere of A_1, H_{a_1} with pole the diametrical point a_1 of the point A_1.

We will denote the open semisphere with pole the point M by H_M.

It is easy to see that:

$$arc(a_1A_j) < \frac{\pi}{2}, forj = 1, 2, 3, 4, 5 \tag{1}$$

From the above (1) we see that the convex sets $H_{A_1}, H_{A_2}, H_{A_3}, H_{A_4}, H_{A_5}$ have every four, non void intersection, so accordly the Helly's theorem for the sphere in E^3,

$$\bigcap_{j=1}^{5} H_{A_j} \neq \emptyset$$

so we can suppose

$$\bigcap_{j=1}^{5} H_{A_j} \neq a$$

Therefore $arc(aA_j) < \frac{\pi}{2}, ..forj = 1, 2, 3, 4, 5$. Hence the H_a includes the five points A_1, A_2, A_3, A_4, A_5. From here we can see that at least one $arc(A_iA_j) \leq \frac{\pi}{2}$. Because of, the points A_2, A_2, A_3, A_4, A_5 can be assigned in the four octant of a Cartesian system $OXYZ$ in three space.

References

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