The perimeters of the cevian and pedal triangle.

George Tsintsifas, Thessaloniki, Greece.

We start with a triangle ABC and an interior point M. The **cevians determined by** M are the line segments AA_1,BB_1,CC_1 through M that join a vertex to a point on the opposite side (with A_1 on BC, B_1, on CA and C_1 on AB). We call $C(M) = \triangle A_1B_1C_1$,**the cevian triangle for M**. The point M also determines a **pedal triangle** $P(M) = \triangle A_2B_2C_2$ whose vertices are the feets A_2,B_2,C_2 of the perpendiculars dropped from M to the sides BC, CA and AB respectively. Problem E2716* in the *American Mathematical Monthly* [1] called for a proof that

$$perimeter C(M) \geq perimeter P(M).$$

C.S. Gardner submitted the only solution; his argument was based on *ad hoc* reasoning in several cases. Some years ago I found a shorter and more analytical proof based on a lemma that seems interesting in its own right.

first Proof

Lemma

Let ABC be a triangle and ϕ, ω, θ three positive convex angles so that $\phi + \omega + \theta = 2\pi$ and M is a point of the plane of the triangle ABC. We denote

$$F(M) = AM.sin\phi + BM.sin\omega + CM.sin\theta$$

case (a). For $\phi \geq A$ $,\omega \geq B,$ $\theta \geq C$ the minimum of F(M) is taken for an internal to ABC point P so that

$$\angle BPC = \phi, \quad \angle CPA = \omega \quad and \ \angle APB = \theta$$

Therefore we will have:

$$F(M) \geq F(P) \tag{1}$$

case (b). For $\phi \leq A$, it holds:

$$AM.sin\phi + BM.sin\omega + CM.sin\theta \geq AB.sin\omega + AC.sin\theta \tag{2}$$

Proof
case (a)
We are refered in an orthogonal Cartesian system O.xyz, and let:
$A = (x_1, y_1), \ \ B = (x_2, y_2), \ \ C = (x_3, y_3), \ \ M = (x, y)$.
We will have:

$$F(M) = F(x, y) = \sum_{i=1}^{i=3} sin\phi\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

cyclic relative $\phi, \ \omega, \ \theta$.
The function Fx,y is positive determined in the triangle ABC, it is continue and has derivates except the vertices A,B,C. We will find the min. of F(x,y) in ABC-(A,B,C) and we will examine it separetely in the vertices A,B,C. Denoting by $e_1, \ e_2$ the unit vectors of the Cartesian systemO.xyz, we see that:

$$gradF(x, y) = \frac{\theta F}{\theta x}.e_1 + \frac{\theta F}{\theta y}.e_2 = \sum_{i=1}^{i=3} sin\phi.\frac{\vec{r_1}}{r_1}$$

cyclic relative $\phi, \ \omega, \ \theta$ and $\vec{r_1} = \vec{AM}, \ \vec{r_2} = \vec{BM}, \ \vec{r_3} = \vec{CM}$. Let now $\frac{\vec{r_1}}{r_1} = a_0, \ \frac{\vec{r_2}}{r_2} = b_0, \ \frac{\vec{r_3}}{r_3} = c_0$. The minimum for F(x,y) is given by:

$$a_0 sin\phi + b_0 sin\omega + c_0 sin\theta = 0 \tag{3}$$

We succesively multiply the relation (1) by $a_0, \ \ b_0, \ \ c_0$. Denoting by $t_1 = b_0.c_0, \ \ t_2 = c_0.a_0 \ \ and \ \ t_3 = a_0.b_0$ we find the system.

$$sin\phi + t_2 sin\omega + t_3 sin\theta = 0$$

$$t_1 sin\theta + sin\omega + t_3 sin\phi = 0$$

$$t_1 sin\omega + t_2 sin\phi + sin\theta = 0$$

The solution of the above system is easy and we take:

$$t_1 = b_0.c_0 = cos(\omega + \theta)$$

That is for the minimum F(x,y) the point M must coincide to a point P, so that

$$\angle BPC = 2\pi - (\omega + \theta) = \phi.$$

Similarly we find that:$\angle CPA = \omega,\ \ \angle APB = \theta.$
We will examine now the case M=A, that is $F(A) = bsin\theta + csin\omega$
Let P the above determined point. We consider the circle BPC of radius R' and we denote by A' the intersection of the line AP and the circle BPC. Ptolemy's inequality to the quadrilateral ABA'C gives:

$$c.CA' + b.BA' \geq (PA + PA').BC = PA.BC + PA'.BC$$

or

$$2R'.c.sin\omega + 2R'.b.sin\theta \geq AP.BC + PA"'.BC$$

From Ptolemy's theorem, we have:

$$PA'.BC = BP.CA' + CP.BA'$$

. From the above and sinus theorem we finaly take $F(A) \geq F(P)$. Similarly $F(B), F(C) \geq F(P)$.
case (b)
The proof of the inequality (2) is elementary but very interesing. Let M be a point of the plane of the triangle ABC. We transform the triangle AMB by a rotation of center A, angle $\pi - A$ and ratio $\frac{sin\omega}{sin\theta}$.
The triangle AMB takes the place AM'A' where C,A,A' are in the line CA with the order C-A-A' (se fig1.).We will have:

$$M'A' = BM.\frac{sin\omega}{sin\theta} \tag{4}$$

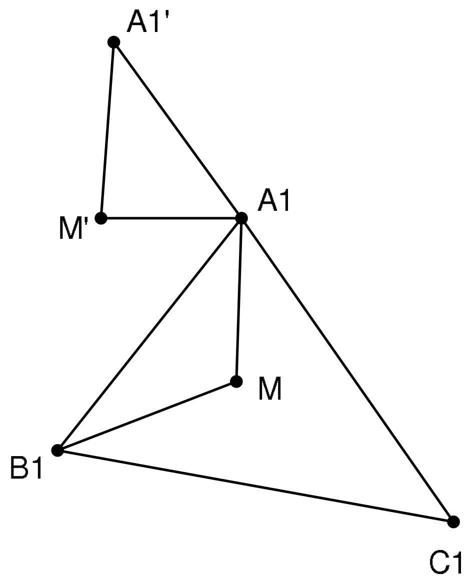

Figure 1:

Also in the triangle MAM' is.

$$\frac{AM'}{AM} = \frac{sin\omega}{sin\theta}, \quad \angle MAM' = \pi - A < \pi - \phi.$$

We construct the triangle PQS so that QP=AM',QS=AM and $\angle PQS = \pi - \phi$. Let $\angle QPS = \theta', \angle QSP = \omega'$. We have:

$$\theta' + \omega' = \phi$$

$$\frac{sin\omega'}{sin\theta'} = \frac{sin\omega}{sin\theta}.$$

From the above equations we find

$$sin(\phi - \theta').sin\theta + sin\theta'.sin(\phi + \theta) = 0$$

and after some manipulations we find:

$$\pi - \theta = \theta' \quad and \quad \pi - \omega = \omega'$$

The triangles MAM', SQP have: AM'=QP, AM=QS and $\angle MAM' = \pi - A < \pi - \phi = \angle PQS$. Therefore

$$MM' < PS = \frac{QS.sin\phi}{sin\theta} = \frac{AM.sin\phi}{sin\theta} \tag{5}$$

From (4) and (5) follows:

$$BM.\frac{sin\omega}{sin\theta} + AM.\frac{sin\phi}{sin\theta} > A'M' + M'M$$

But,

$$A'M' + M'M + CM > AA' + AC$$

From the above two inequalities we take.

$$AM.\frac{sin\phi}{sin\theta} + BM.\frac{sin\omega}{sin\theta} + CM > AB\frac{sin\omega}{sin\theta} + AC.$$

Theorem

For every triangle ABC and an interior point M, the perimeter of the cevian triangle is bigger or equal to the perimeter of its pedal triangle.

Proof

Let $A_1B_1C_1$ the cevian triangle and $A_2B_2C_2$ the pedal triangle of the point M, see fig.2). It is well known that the circles $p_1 : B_1AC_1,\ p_2 : C_1BA_1,\ p_3 : A_1CB_1$ have a common point P (Miquel's point, see [2]). We denote R_1, R_2, R_3 the radii of p_1, p_2, p_3 respectively. We easily see that:

$$perimeterC(M) = \sum B_1C_1 = \sum 2R_1 \sin A \tag{6}$$

$$perimeterP(M) = \sum B_2C_2 = \sum AM \sin A \tag{7}$$

The meaning of the sums are easily understood.

case 1. We suppose that $\angle B_1PC_1 = \pi - A > \angle B_1A_1C_1,\quad \angle C_1PA_1 = \pi - B > \angle C_1B_1A_1,\quad \angle A_1PB_1 = \pi - C > \angle A_1C_1B_1$, that is P is an interior point of the triangle $A_1B_1C_1$.

We have:

$$PA + PA_1 \geq AM + MA_1$$

or

$$2R_1 + PA_1 \geq AM + MA_1$$

and we see that:

$$\sum 2R_1 \sin A + \sum PA_1 \sin A \geq \sum AM \sin A + \sum MA_1 \sin A \tag{8}$$

In this point we use the lemma for the triangle $A_1B_1C_1$. We know that :

$$\angle B_1PC_1 = \pi - A, \angle C_1PA_1 = \pi - B, \angle A_1PB_1 = \pi - C$$

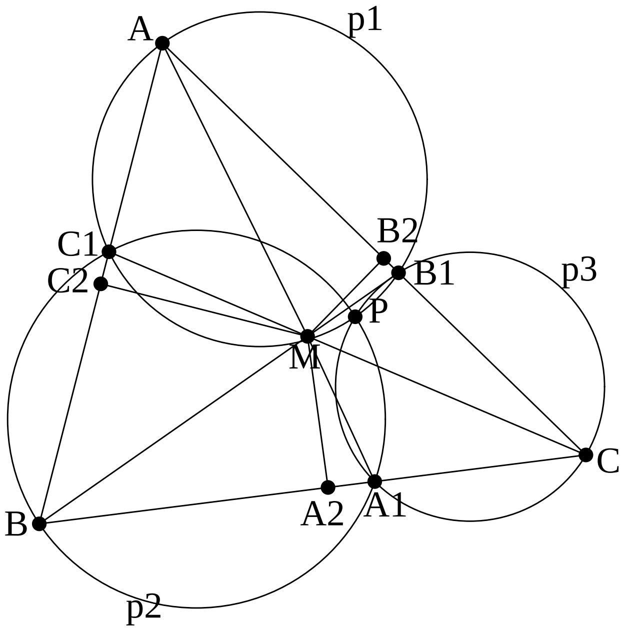

Figure 2:

.
Therefore:

$$\sum MA_1 \sin A \geq \sum PA_1 \sin A \qquad (9)$$

From (8), (9) we see that:

$$\sum 2R_1 \sin A \geq \sum AM \sin A$$

that is from (6),(7) we conclude

$$perimeterC(M) \geq perimeterP(M)$$

.
The equality case for P=M, that is C(M)=P(M) or M=H the orthocenter.

case 2..We suppose now that $\angle B_1PC_1 = \pi - A < \angle B_1A_1C_!$. In this case is $A + \angle B_1A_1C_1 > \pi$, therefore the point A_1 is an interior point of the circle B_1AC_1.
The following inequalities are obvius.

$$2R_1 \geq AM + MA_1 \tag{10}$$

$$2R_2 + A_1B_1 \geq A_1B + A_1B_1 \geq BM + MB_1 \tag{11}$$

$$2R_3 + A_1C_1 \geq A_1C + A_1C_1 \geq CM + MC_1 \tag{12}$$

We multiplay the above respectively by $sinA, sinB, sinC$. Adding, from the lemma **case (b).**, we find again that $perimeterC(M) \geq perimeterP(M)$.

second Proof

Lemma
Let $A_1B_1C_1$ be a triangle and ϕ,ω,θ are three positive convex angles so that:

$$\phi + \omega + \theta = 2\pi. \tag{13}$$

For an interior point M the minimum of

$$Q(M) = A_1M \cdot \sin\phi + B_1M \cdot \sin\omega + C_1M \cdot \sin\theta \tag{14}$$

is taken
case (a):
For $\phi \geq \angle B_1A_1C_1, \;\; \omega \geq \angle C_1B_1A_1, \;\; \theta \geq \angle A_1C_1B_1$.

in the unique position of the point M=P so that:

$$\angle B_1PC_1 = \phi, \quad \angle C_1PA_1 = \omega, \quad \angle A_1PB_1 = \theta.$$

Therefore it holds: $Q(M) \geq A_1P \cdot \sin\phi + B_1P \cdot \sin\omega + C_1P \cdot \ sin\theta$.
case (b):
For $\phi < \angle B_1A_1C_1$, it holds:

$$Q(M) \geq B_1A_1 \cdot \sin\omega + C_1A_1 \cdot \sin\theta$$

Proof of the case (a).

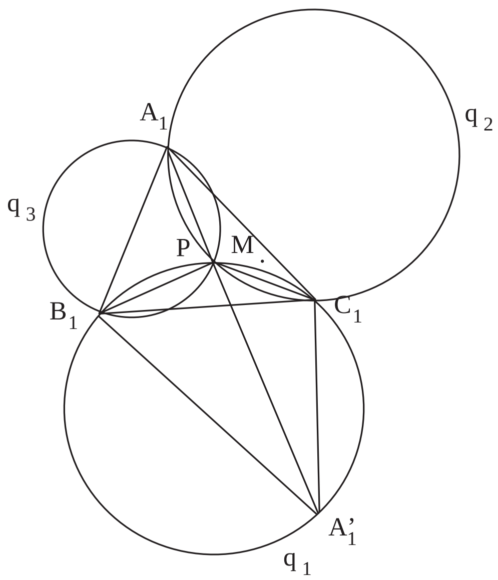

Figure 3:

We accept the following notation. The circle which in a chord LY accepts an angle equal to θ is denoted by: (LY, θ).
We consider the circles q_1:(B_1C_1, ϕ), q_2:(C_1A_1, ω), q_3:(A_1B_1, θ). The circles have a common point P, see fig. 3. So we have: $\angle B_1PC_1$=ϕ, $\angle C_1PA_1$=ω, and $\angle A_1PB_1$=θ. The line A_1P intersects q_1 at the point A_1'. We easily see that:

$$\angle B_1A_1'C_1 = \pi-\phi, \angle C_1B_1A_1' = \angle A_1'PC_1 = \pi-\omega, \angle A_1'C_1B_1 = \angle A_1'PB_1 = \pi-\theta \tag{15}$$

Ptolemy's theorem for the inscribed quadrilaterar $PB_1A_1'C_1$ gives:

$$PA_1' \cdot B_1C_1 = PB_1 \cdot C_1A_1' + PC_1 \cdot B_1A_1'. \tag{16}$$

From (15),(16) and the sinus theorem applied to the triangle $A_1'B_1C_1$ we get:

$$PA_1' \cdot \sin\phi = PB_1 \cdot \sin\omega + PC_1 \cdot \sin\theta \tag{17}$$

For every point M not belonging to q_1, Ptolemy's inequality asserts:

$$MA_1' \cdot B_1C_1 < MB_1 \cdot C_1A_1' + MC_1 \cdot BA_1' \tag{18}$$

or,

$$MA_1' \cdot \sin\phi < MB_1 \cdot \sin\omega + MC_1 \cdot \sin\theta \tag{19}$$

Supposing that M does not belong to A_1A_1', we see that:

$$A_1M + MA_1' > AP + PA_1' \tag{20}$$

or from (17),(18),(19) and (20) follows:

$$A_1M + B_1M\frac{\sin\omega}{\sin\phi} + C_1M\frac{\sin\theta}{\sin\phi} > A_1P + B_1P\frac{\sin\omega}{\sin\phi} + C_1P\frac{\sin\theta}{\sin\phi}. \tag{21}$$

and our lemma **case (a)** is proved. Obviously we have equality for $M = P$.
Proof of the **case (b)**.
Let M be a point of the triangle $A_1B_1C_1$. We transform the triangle A_1MB_1 by a rotation of center A_1, angle $\pi - \angle B_1A_1C_1$ and ratio $\frac{sin\omega}{sin\theta}$.
The triangle A_1MB_1 takes the place $A_1M'A_1'$ where C_1, A_1, A_1' are on the line C_1A_1 in the order $C_1 - A_1 - A_1'$ (se fig4.).We have:

$$M'A_1' = B_1M\frac{\sin\omega}{\sin\theta} \tag{22}$$

Also in the triangle MA_1M' occurs.

$$\frac{A_1M'}{A_1M} = \frac{\sin\omega}{\sin\theta}, \quad \angle MA_1M' = \pi - A_1 < \pi - \phi.$$

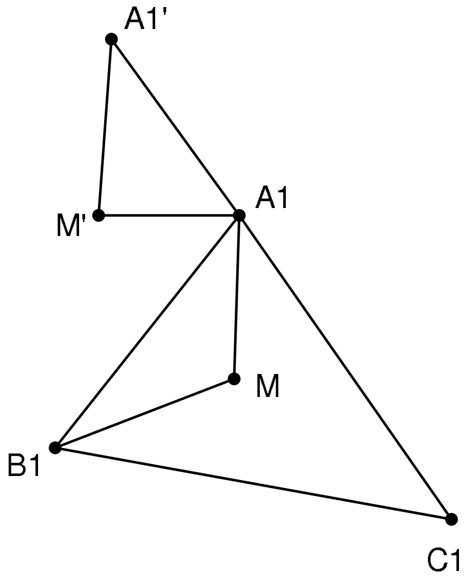

Figure 4:

We construct the triangle PQS so that $QP = A_1M'$, $QS = A_1M$ and $\angle PQS = \pi - \phi$.
Let $\angle QPS = \theta', \angle QSP = \omega'$. We have:

$$\theta' + \omega' = \phi,$$

$$\frac{\sin\omega'}{\sin\theta'} = \frac{\sin\omega}{\sin\theta}.$$

From the above equations we find

$$\sin(\phi - \theta') \cdot \sin\theta + \sin\theta' \cdot \sin(\phi + \theta) = 0$$

and after some elementary calculations we take:

$$\pi - \theta = \theta' \quad and \quad \pi - \omega = \omega'$$

The triangles MA_1M', SQP we have: $A_1M' = QP$, $A_1M = QS$ and $\angle MA_1M' = \pi - A_1 < \pi - \phi = \angle PQS$. Hence

$$MM' < PS = \frac{QS \cdot \sin\phi}{\sin\theta} = \frac{A_1M \cdot \sin\phi}{\sin\theta}. \tag{23}$$

From (22) and (23) follows:

$$B_1M\frac{\sin\omega}{\sin\theta}+A_1M\frac{\sin\phi}{\sin\theta}>A_1'M'+M'M$$

and together with

$$A_1'M'+M'M+C_1M>A_1A_1'+A_1C_1$$

we get.

$$A_1M\frac{\sin\phi}{\sin\theta}+B_1M\frac{\sin\omega}{\sin\theta}+C_1M>A_1B_1\frac{\sin\omega}{\sin\theta}+A_1C_1.$$

Theorem

For every triangle ABC and an interior point M, the perimeter of the cevian triangle is not less than the perimeter of its pedal triangle.

Proof

Let $A_1B_1C_1$ the cevian triangle and $A_2B_2C_2$ the pedal triangle of the point M, see fig. 5. It is well known that the circles $p_1 : B_1AC_1,\ \ p_2 : C_1BA_1,\ \ p_3 : A_1CB_1$ have a common point P (Miquel's point, see [2]). We denote by R_1, R_2, R_3 the radii of p_1, p_2, p_3 respectively and by A, B, C the angles of the triangle ABC. We easily see that:

$$perimeterC(M)=\sum B_1C_1=\sum 2R_1\cdot\sin A \tag{24}$$
$$perimeterP(M)=\sum B_2C_2=\sum AM\cdot\sin A$$

The meaning of the sums is easily understood.

case 1.

We suppose that $\angle B_1PC_1=\pi-A>\angle B_1A_1C_1,\quad \angle C_1PA_1=\pi-B>\angle C_1B_1A_1,\quad \angle A_1PB_1=\pi-C>\angle A_1C_1B_1$, that is P is an interior point of the triangle $A_1B_1C_1$.

We also have:

$$PA+PA_1\geq AM+MA_1$$

or

$$2R_1+PA_1\geq AM+MA_1$$

and we see that:

$$\sum 2R_1\cdot\sin A+\sum PA_1\cdot\sin A\geq\sum AM\cdot\sin A+\sum MA_1\cdot\sin A \tag{25}$$

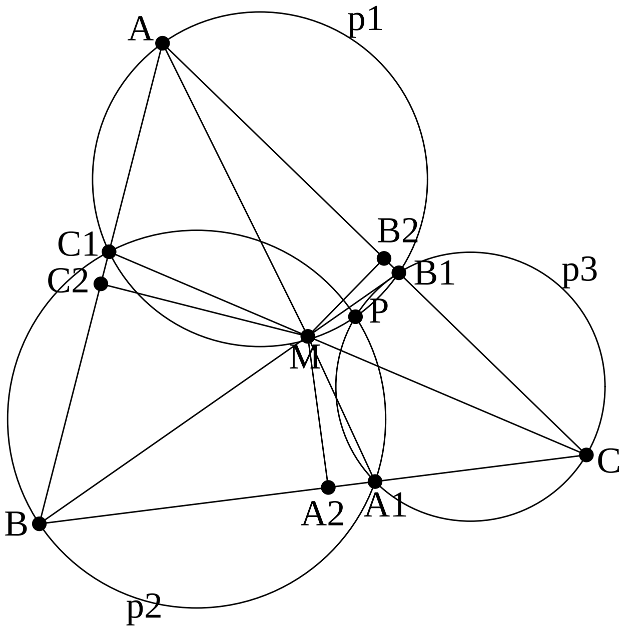

Figure 5:

Now we apply **case (a)** of the lemma to the triangle $A_1B_1C_1$.
We know that :

$$\angle B_1PC_1 = \pi - A, \angle C_1PA_1 = \pi - B, \angle A_1PB_1 = \pi - C.$$

and thus

$$\sum MA_1 \cdot \sin A \geq \sum PA_1 \cdot \sin A \tag{26}$$

From (25), (26) we see that:

$$\sum 2R_1 \cdot \sin A \geq \sum AM \cdot \sin A$$

that is from (24) we conclude

$$perimeterC(M) \geq perimeterP(M)$$

.
The equality case for P=M, that is C(M)=P(M) or M=H the orthocenter.

case 2.
We suppose now that $\angle B_1PC_1 = \pi - A < \angle B_1A_1C_!$. In this case is $A + \angle B_1A_1C_1 > \pi$,and therefore the point A_1 is an interior point of the circle B_1AC_1.
The following inequalities are obvius.

$$2R_1 > AM + MA_1 \tag{27}$$

$$2R_2 + A_1B_1 \geq A_1B + A_1B_1 > BM + MB_1 \tag{28}$$

$$2R_3 + A_1C_1 \geq A_1C + A_1C_1 > CM + MC_1 \tag{29}$$

We respectively multiply the above by $\sin A, \sin B, \sin C$ and then we add them up. From **case (b)** of the lemma, we find again that $perimeterC(M) > perimeterP(M)$.

Remarks

The question which arises, after the solution of the above problem, is about the relation of the area between C(M) and P(M). The remark (1) gives the answer; that is, there are points M so that the AreaC(M) is bigger than the AreaP(M) and for other points holds the converse. Probably, it would be of some interest to determine the points M so that: AreaC(M)=AreaP(M).

1. Well known inequalities about the area of C(M) and P(M) are:

$$AreaC(M) \leq \frac{1}{4} AreaABC$$

see [3].

$$AreaP(M) \leq \frac{1}{4} AreaABC$$

see [4].
Also we obviously have

$$AreaC(O) \leq AreaP(O) = \frac{1}{4}AreaABC$$

$$\frac{1}{4}AreaABC = AreaC(G) \geq AreaP(G)$$

where O and G are the circumcenter and the centroid of ABC.
2. Our lemma can be considered as an extension of Fermat-Steiner theorem , see [2], about the minimum of the sum AP+BP+CP. Indeed for $\phi = \omega = \theta = \frac{2}{3}\pi$ we have the Fermat-Steiner point.

References

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George Tsintsifas
23 Platonos str.
54631 Thessaloniki
Greece
e.mail: gtsintsifas@yahoo.com