Some Inequalities for convex sets

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A former student of mine, E. Symeonidis, has sent to me some interesting problems about Convexity. The problems were published in Axioms 2018 7(1) by S. Marcus and F. Nichita. I think that I have some results about.

Problems

Let f be a convex figure in the plane (that is compact convex set). We denote by G its centroid. D is the maximal chord and d the minimal chord through G. Also stands L for the perimeter, D_F the diameter, d_F the minimal breadth, and A the area of F.

We have to prove:

$$(a) \quad L \geq d\pi.$$

$$(b) \quad d.D > A.$$

$$(c) \quad L.D \geq 4A.$$

Proofs.

Inequality (a).

The formula for the perimeter of a convex figure F see [1],[2] is:

$$L = \frac{1}{2}\int_0^{2\pi} B(\vartheta)d\vartheta$$

where $B(\vartheta)$ is the breadth of F to the direction ϑ, d_F is the min. breadth of F. So we wll have:

$$L \geq \frac{1}{2}\int_0^{2\pi} d_F.d\vartheta \geq d\pi.$$

That is because of the obvious $d_F \geq d$
The equality for the circle and the convex figures with constant breadth.

For (b) and (c) we need two lemmas.

1. lemma.
In the side AB of a parallelogram $ABCD$ we have the points E, Z so that: $EZ = \frac{1}{2}AB$. We take the points $M \in [AD]$ and $N \in [BC]$ and $P = ME \cap NZ$ we will prove that:

$$Area(MAE) + Area(NBZ) \geq Area(EPZ) \qquad (1)$$

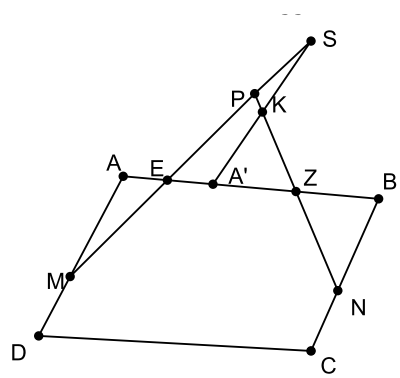

Fig. 1

Proof
We take $EA' = EA$. Obviously $ZA' = ZB$. We suppose $AE < BZ$ or $A'E < A'Z$, hence the parallel from A; to AD intersects the str.line segmant to the point $K \in [PZ]$. Therefore $Area(MAE) = Area(EA'S$ and $Area(NBZ) = Area(A'ZK)$. So (1) follows.

2. lemma.

In the perimeter ϑF of the convex set F there are the points A, B, A', B'. The chord AB and $A'B'$ are parallel and the point $P = AA' \cap BB'$ is outside of F. We denote by $arcAB = c, arcA'B' = c'$ on the ϑF and $c \supset c'$, like in the fig.2. We will prove that:

$$\frac{L(c)}{|A-B|} \geq \frac{L(c')}{|A'-B'|}$$

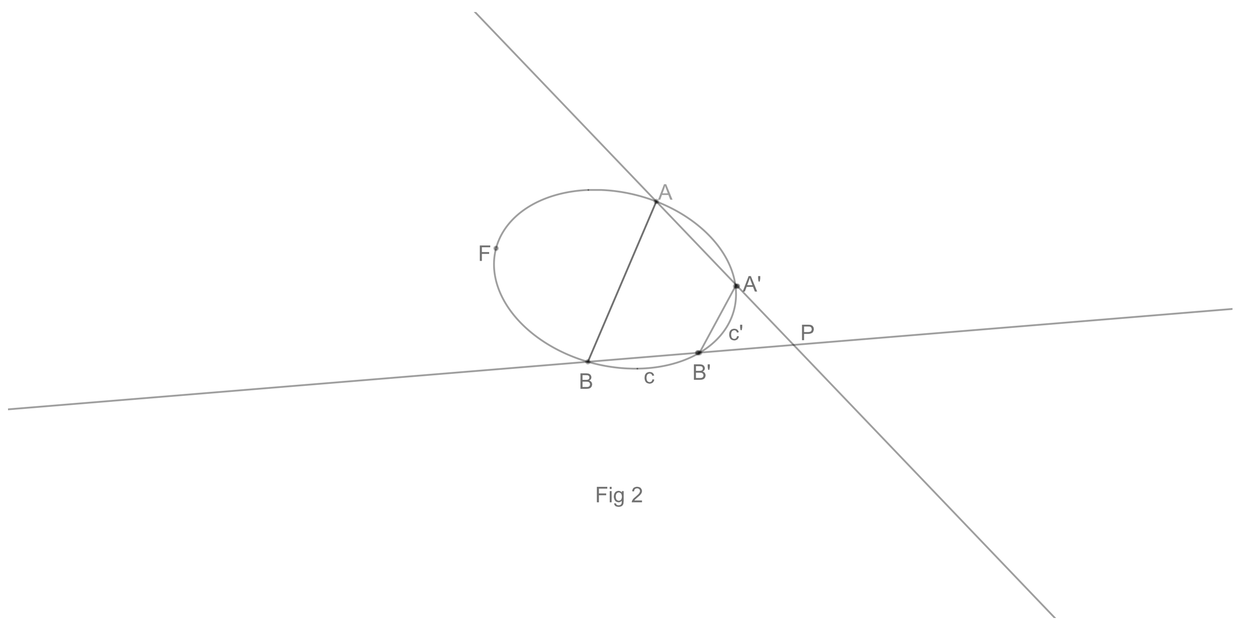

Fig 2

Proof

We denote by M the vector $\vec{O}M$.

We have $A - B = \mu(A' - B')$ We easily find

$$P = \frac{\mu A' - A}{\mu - 1} = \frac{\mu B' - B}{\mu - 1}$$

hence

$$P - A' = \frac{A' - A}{\mu - 1} \qquad (1)$$

and

$$P - B' = \frac{B' - B}{\mu - 1} \qquad (2)$$

but

$$L(c) \geq |A - A'| + |B - B'| + L(c') \qquad (3)$$

From the above (1),(2) follows

$$|A - A'| + |B - B'| = |\mu - 1|(|P - A'| + |P - B'|) \geq |\mu - 1|L(c') \qquad (4)$$

From (3) and (4) we take

$$L(c) \geq |\mu - 1|L(c') + L(c') = \mu L(c')$$

and finally

$$L(c) \geq \frac{|A - B|}{|A' - B'|}L(c')$$

Now the **inequality (b)**.
The continuity of the convexity asserts us that we can choose the diametrical chord AB so that:

$$d_F \leq AB \leq D \leq D_F \qquad (5)$$

Where w_F and D_F stands for the min.breadth and diameter of F respectively.

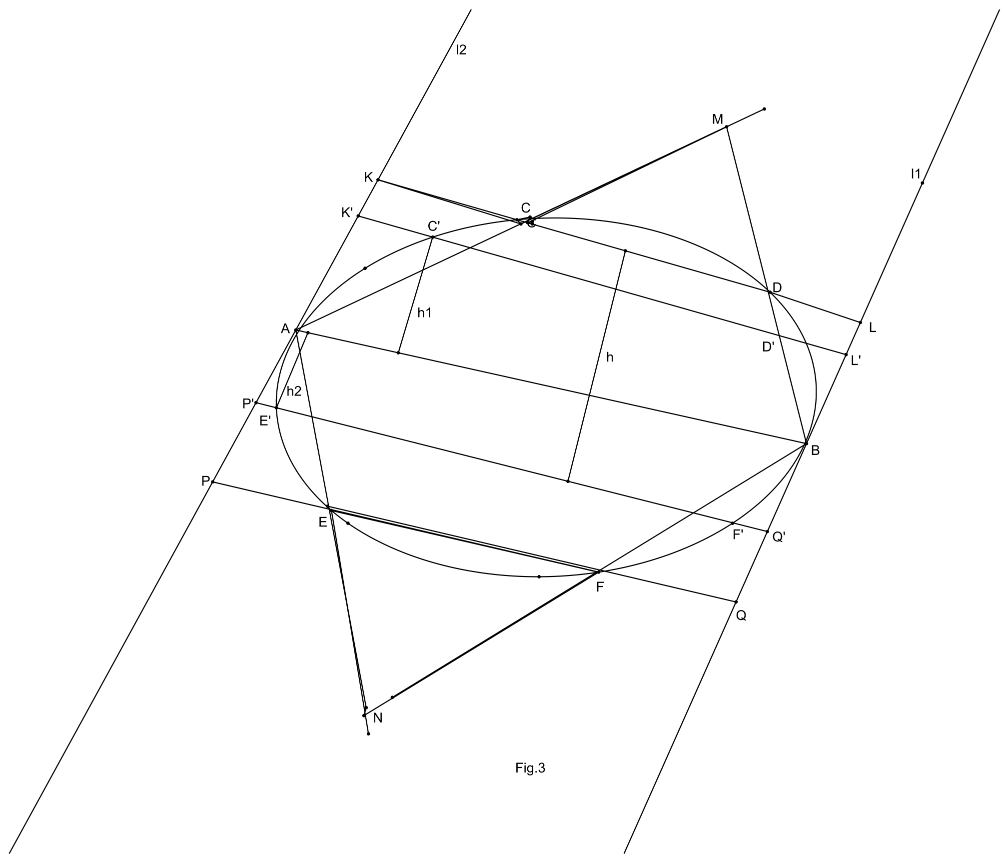

Fig.3

See fig.3
l_1 and l_2 are the parallel supporting lines at the points A and B
We take on $\vartheta F \quad CD = EF = \frac{AB}{2}, and \quad CD \| EF \| AB.$
We easily see, according our first lemma that

$$Area(AKC) + Area(BLD) \geq Area(CMD)$$

That means

$$Area(F) < Area(KLQP) = AB.h \qquad (6)$$

where h is the distance between CD, EF.
We accept that we can choose AB so that the center of $KLQP$ and the centroid of F, be close enough, so $d \geq h$, and from the above (6) follows that

$Area(F) < D.d$

Inequality (c). The equality only for F circle. We suppose that F is not a circle.
We translate the str. lines KL, PQ closer towards to AB until to $K'L', P'Q'$ such a way the parallelogram has

$$Area(K'L'Q'P') = Area(F) \qquad (7)$$

we understand that we have:

$$C'D' > CD, \qquad F'E' > FE$$

where $(C', D') = K'L' \cap F$ and $(F', E)' = P'Q' \cap F$
Let now L_1 the part of the perimeter L over of AB and analogously L_2. We see that $C'D'/AB > 1/2$ and $E'F'/AB > 1/2$. So, from the lemma 2, easily see that $arcC'D' > L_1/2$ and $arcE'F' > L_2/2$. That is $arcC'D' + arcE'F' > L/2$
From the above we conclude that $L/2 > arcC'E' + arcD'F'$ but $arcC'E' + arcD'F' > 2h'$ where h' is the distance of the parallel lines $K'L', P'Q'$. so from the above and (7) we take:
$Area(F) = AB.h'$ but from (5) have $D \geq AB$ and from the above $L/4 > h'$ we finally find $L.D > 4Area(F)$.

Notable Comment:
In the proof we did'n't use the properties of the centroid G. This means that the Inequalities are correct for every point P instead of G. For the inequality (b), we have in mind, our remark about the position of the point P.

References:
1.Theory of Convex bodies,T. Bonnesen and W. Fencel, B. Associates
2.Convex figures, G.Tsintsifas (My side, " gtsintsifas" page 14)
3. Convex figures, I.M. Yaglom, V.G.Boltyanskii, Holt,Rinehart and Winston