The inscribed parallelepiped in convex body in E^3

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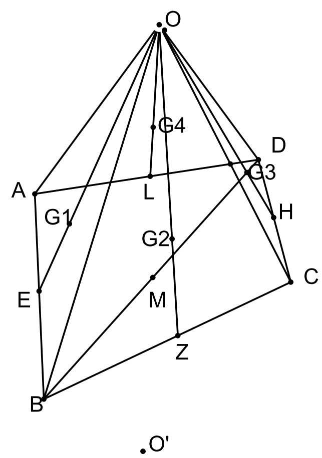
Well known theorems from Analysis assert that there is a circumscribed polygon P with n sides of minimum area around of a convex figure K in E^2 and a circumscribed polyhedron F of minimum volume around of convex body K in E^n. From Day's Theorem see [1] we know that the middle points of P are in the boundary of K and for the polyhedron F the cendroids of the faces must be on the boundary of K. The minimum quadrilateral P circumscribed around the convex figure K has the middle poinds of the sides on K and easily we see that these points are the vertices of a parallelograme P_1 and $areaP_1 = \frac{1}{2}areaP \geq \frac{1}{2}areaK$. In this paper we will prove an analogous theorem for E^3, that is:

Theorem 1.
There is an inscribed parallelepiped F_1 in convex body K in E^3 so that:

$$volumeF_1 \geq \frac{2}{9}.volumeK \tag{1}$$

Proof
From Day's paper [1] we know that the Octahedron $F = OABCDO'$ of minimum volume circumscribed to K has its centroids $G_1, G_2, G_3, G_4, -G_1,' G_2,' G_3,' G_4'$ of the edges $OAB, OBC, OCD, ODA, O'AB, O'BC, O'CD, O'DA$ on K. We also denote $P_1 = G_1G_2G_3G_4G_1'G_2'G_3'G_4'$ and it is elementary to see that P_1 is parallilepiped (the sides of $G_1G_2G_3G_4$ are parallel to BD, AC etc.) see fig.



We will prove that

$$VolumeF_1 = \frac{2}{9} VolumeF \qquad (2)$$

Let M the middlepoint of BD and h the distance between the planes $G_1G_2G_3G_4$ and $G'_1G'_2G'_3G'_4$. By h_0 we denote the unit vector perpendicular to the plane $G_1G_2G_3G_4$. Also the altitudes of the pyramides $OABD$ and $O'ABD$ are h_1, h'_1 and the perpendicular unit vector to the plane ABD is h_{10}. For the pyramides $OBCD, O'BCD$ are h_2, h'_2 and h_{02} respectively. We see now that the vectors h_0, h_{10}, h_{2o} are perpendicular to the line BD so they are conplanar. Supposing the non obtuse angle of the two planes ABD and CBD equal to 2θ we are leading to the following relations.

$$3h.cos\theta = h_1 + h'_1 = h_2 + h'_2 \qquad (3)$$

$$[EZHL] = [EBML]cos\theta + [ZBMH]cos\theta \qquad (4)$$

where E, Z, H, L the middlepoints of AB, BC, CD, DA respectively and the braquets denote area. To explain the relation (3) we say that follows projecting h, h_1, h_2 to a line perpendicular to the plane $G_1G_2G_3G_4$ and the (4) projecting $EBML, ZBMH$ to the plane $G_1G_2G_3G_4$.
Also

$$VolF_1 = [G_1G_2G_3G_4].h = \left(\frac{2}{3}\right)^2 [EZHL].h$$

From (3),(4) follows that

$$VolF_1 = \left(\frac{2}{3}\right)^2 . \frac{1}{3}[EBML](h_1 + h_1') + \left(\frac{2}{3}\right)^2 . \frac{1}{3}[ZBMH](h_2 + h_2']$$

or

$$VolF_1 = \left(\frac{2}{3}\right)^2 . \frac{1}{6}[ABD](h_1 + h_1') + \left(\frac{2}{3}\right)^2 . \frac{1}{6}[BCD](h_2 + h_2']$$

and finally

$$VolF_1 = \frac{2}{9} Vol(OABCDO') \geq \frac{2}{9} Vol.K$$

The equality for the octahedron.

Let us see the case of the $K=$ triangle in the plane and $K=$tetrahedron in E^3. For the triangle ABC with middlepoints of the sides D, E, F, obviously $Area(AFDE) = \frac{1}{2}Area(ABC)$.The case of tetrahedron $OABC$ is more complicate. Let G_1, G_2, G_3 the cendroits of the triangles OAB, OBC, OCA . The parallels from G_1, G_2, G_3 to the line OA intersect $AB, planeABC, CA$ respectively at the points G_1', G_2', G_3'. The volume of the parallelepiped $A'G_1G_2G_3AG_1'G_2'G_3'$ is $\frac{2}{9}Vol(OABC)$, where A' is the intersection of the plane $G_1G_2G_3$ with the line OA.
A very interesting consequence from the theorem.1 follows using an inequality of G.D. Chakerian see [3]. The Chakerian 's theorem is about a convex body K in E^n, an iscribed parallelepiped F_1 in K and a circumscribed F_0 parallelepiped to K with faces parallel to F_1. That is:

$$(VolK)^n \geq VolF_0 . \Big(VolF_1\Big)^{n-1}$$

According the Chakerian's inequality we take:

$$VolF_0 \leq \left(\frac{9}{2}\right)^2 .VolK \tag{5}$$

Theorem 2.
There is an circumscribed parallelepiped F_1 to convex body K in E^3 so that:

$$volumeF_0 \leq \left(\frac{9}{2}\right)^2 volumeK \tag{6}$$

Comment. The above problem in E^n , seems quite complicate, but we have to conjecture that for a convex body K in E^n there is an inscribed parallelepiped F_1 so that

$$Vol(F_1) \geq \frac{n!}{n^n} vol(K)$$

and for the circumscribed parallelepiped using the Chakerian'sinequality an analogous to (2) theorem.

References

1.Mahlon M. Day. Polygons circuscribed about closed convex curves. Trans.Am.Math.Soc. Vol 62 pp 315-319, 1957.
2.T.Hausel, E.Makai, A.Szucs, Polyhedra inscribed and circumscribed to convex bodies.
3.G.D. Chakerian,Elemente der Math. 1973,108.