Max. and Min. problems for the Tetrahedron.

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One of the most famous problem in Geometry is the isoperimetric problem, that is:
Of all plane simple (wihtout double points) closed curves with the same length, which one has the max. area.
Several ingenious proofs (especially Steiner's proof) appeared. At first was felt that the problem was solved, later it was understood that the existance problem had been omited. The theory of Convex sets contributed a satisfactory solution to many problems of this kind.
The solution for the general isoperimetrc problem in E_n is given by the inequality

$$\left(\frac{S}{\omega_n}\right)^n \geq \left(\frac{V}{k_n}\right)^{n-1}$$

Where S the "perimeter" and V the volume of the body F, k_n the volume and ω_n the "perimeter" of the unit sphere in E^n.
For E_3 the inequality is

$$\left(\frac{S}{4\pi R^2}\right)^3 \geq \left(\frac{V}{4/3\pi R^3}\right)^2$$

and for E^2

$$S^2 \geq 4\pi V$$

The problem of determing among all polyhedra which one has the max volume for a given surface is difficult and is solved for only some cases.
In this paper we will try to give some answers for the tetrehdron and will see the problem of the max. and min. of the tetrahedron circumscribed on a sphere and inscribed in a sphere.

Proposition 1
The tetrahedron $ABCD$ has constand volume $V = Q^3$ and constand basis BCD. The minimum of the surface $S = (ABC) + (ACD) + ADB)$ is taken, when the projection of the vertex A to the triangle BCD is the incenter of that triangle.
Proof
Let AK the perpendicular from A to the plane BCD. It is easy to see that the point K will be inside of the triangle BCD. Therefore, according the standard theorems of Analysis, see [2], the min.S exists.
We drop KE, KZ, KH the perpediculars from K to the sides BC, CD, DB of the triangle BCD respectively. We see that AE, AZ, AH are the altitudes of the triangles ABC, ACD, ADB respectively. So we will have:

$$2S = (ABC) + (ACD) + (ABD) = AE.BC + BZ.CD + AH.BD$$

For simplicity we will denote:

$$AK = h, KE = a_1, KZ = a_2, KH = a_3, BC = b_1, CD = b_2, DB = b_3$$

Therefore

$$2S = b_1\sqrt{h^2 + a_1^2} + b_2\sqrt{h^2 + a_2^2} + b_3\sqrt{h^2 + a_3^2}$$

or

$$2S = \sum_{i=1}^{3} \sqrt{h^2 b_i^2 + a_i^2 b_i^2}$$

From Minkowski inquality have:

$$2S \geq \sqrt{h^2(b_1 + b_2 + b_3)^2 + (a_1 b_1 + a_2 b_2 + a_3 b_3)^2} \tag{1}$$

Hence

$$2S \geq \sqrt{h^2(b_1 + b_2 + b_3)^2 + 4(BCD)^2}$$

Therefore

$$min.S = \frac{1}{2}\sqrt{h^2(b_1 + b_2 + b_3)^2 + 4(BCD)^2}$$

The condition for the equality, according Minkowski inequality is:

$$\frac{hb_1}{a_1 b_1} = \frac{hb_2}{a_2 b_2} = \frac{hb_3}{a_3 b_3} \tag{2}$$

That is $a_1 = a_2 = a_3$. So the point K must be the incenter of BCD. From the above we conclude that the triangles AKE, AKZ, AKH are equal , hence the dihedral angles BC, CD, DB are equal.
Now using contradiction we can prove that all the dihedral of the min.S tetrahedron must be equal. It a simple problem to prove that the tetrahedron with all the dihedrons equal is the regular. Suppose that A_1, B_1, C_1, D_1 are the common points of the insphere with the facets. From the equal dihedral angles we see that $A_1B_1C_1D_1$ is a regular tetrahedron and obviosly the $ABCD$ is regular. Accordingly as immediate consequance is the

Theorem 1

From the tetrahedrons with the same volume, the min. surface has the regular.

Theorem 2

From the tetrahedrons with the same surface F the max. volume has the regular.
In this proof and in next proofs we can accept for the existance problem the standart theorems from the Analysis. Now, let $A_1B_1C_1D_1$ is a non regular tetrahedron with constand surface F and volume V. From the Thorem 1 follows that we can find a tetrahedron $A_0B_0C_0D_0$ with the same volume V and a bigger surface than $A_1B_1C_1D_1$. That is a contradiction.

The next problem is about the min.volume of the tetrahedrons circumscribed to a given sphere. For the solution we wll need three propositions.

Proposition 2

The point P is in the interior of the trihedral angle $AXYZ$. The plane through the point P , intersecting from the trihedral a tetrahedron $ABCD$ of min. volume, has as barycenter of the triangle BCD the point P.

Proof

Let $ABCD$ the tetrahdron so that: B, C, D are in AX, AY, AZ respectively and p_1, p_2, p_3 the distances of P from the planes ACD, ABD, ABC respectively. Also

$$AB = l_1, AC = l_2, AC = l_3, CD = b_1, DB = b_2, BC = b_3,$$

$$(ACD) = a_1, (ADB) = a_2, (ABC) = a_3, (ABCD) = V$$

We will have:

$$3V = a_1p_1 + a_2p_2 + a_3p_3$$

From AM-GM inequality, follows that

$$V \geq \sqrt[3]{a_1a_2a_3p_1p_2p_3} \tag{3}$$

The equality when:

$$a_1p_1 = a_2p_2 = a_3p_3 \tag{4}$$

The above (4) gives the qualitative solution of the problem. That is the min.V is optained from the equality of the volumes $(PACD), (PADB), (PABC)$. We easily can find the above min.$Vol(ABCD)$, but we need the following simple lemma

lemma

For the trihedral angle $AXYZ$ and the B, C, D on the lines AX, AY, AZ holds:

$$\frac{Vol(ABCD)}{AB.AC.AD} = k \quad constand. \tag{5}$$

Indeed, for B_0, C_0, D_0 constand points on AX, AY, AZ we have:

$$\frac{Vol(ABCD)}{AB.AC.AD} = \frac{Vol(AB_0C_0D_0)}{AB_0.AC_0.AD_0} = k \quad constand$$

We denote $sin\angle CAD = k_1, \quad sin\angle DAB = k_2, \quad sin\angle BAC = k_3$ so from (4) and (5) follows

$$k_3p_3l_1l_2 + k_1p_1l_2l_3 + k_2p_2l_1l_3 = l_1l_2l_3.k_1$$

therefore

$$\frac{k_1p_1}{l_1} + \frac{k_2p_2}{l_2} + \frac{k_3p_3}{l_3} = k \tag{6}$$

Also from (4)

$$l_2 = \frac{l_1k_2p_2}{k_1p_1}, \quad l_3 = \frac{l_1k_3p_3}{k_1p_1} \tag{7}$$

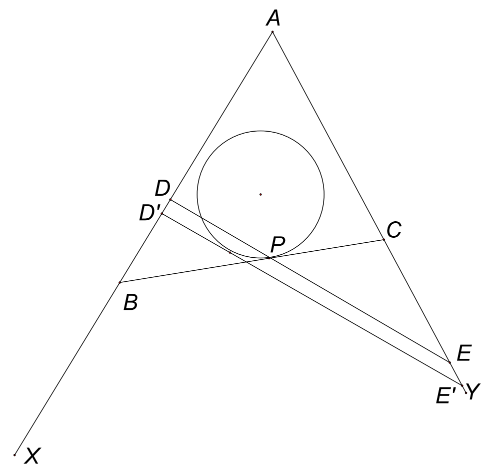
From (6),(7) we take

$$l_1 = \frac{3k_1p_1}{k}, \quad l_2 = \frac{3k_2p_2}{k}, \quad l_3 = \frac{3k_3p_3}{k}$$

That is the triangle BCD is determined and the volume min.V from (3) follows.

$$min.V = 9\frac{p_1p_2p_3}{k^2}\sqrt[3]{k_1^2k_2^2k_3^2}$$

Proposition 2
Inside of the trihedral angle $AXYZ$ there is a sphere (O, R) . A tangent plane p to the sphere at a point P intersects the lines AX, AY, AZ at the points B, C, D respectively. The min.of the volume of the tetrahedron $ABCD$ is taken for $P=$ the centroid of the triangle BCD.
The proof is easy and is the same for the plane.



Let BC the tangent to the circle (c) at the point P so that $BP = PC$ (in E^3) P will be the centroid that is the middle point of BC. The triangle ABC is the min. A second tangent to the circle intersects te sides AX, AY at the points D', E' respectively. We drow through P the parallel to $D'E'$ which intersects AX, AY at the points D, E. We have.$(ABC) \leq (ADE) \leq (AD'E')$ From the above, using contradiction (reductio ab absoredium) we can prove that if a tetrahedron circumscribed to a sphere (O, r) has the tangents points the centroid of the faces, then the tetrahedron is regular.
We need the Euler's inequality. For the tetrahedron holds:

$$R \geq 3r, \quad where \quad r = inradius, \qquad For \quad E^n \quad R \geq nr$$

For the proof it is enough to take the pericenter as interior point of the tetrahedron.
Let h_a, h_b, h_c, h_d the altitudes of the tetrahedron $ABCD$ and OA', OB', OC', OD' the distances of the point O from the faces BCD, ACD, ABD, ABC. Obviously have:

$$R + OA' \geq h_a$$
$$R + OB' \geq h_b$$
$$R + OC' \geq h_c$$
$$R + OD' \geq h_d$$

We multiplay the above inequalities succesively by $(BCD), (ACD), (ABD, (ABC)$ and summing the inequalities we find.

$$SR + 3V \geq 12V$$

or

$$SR \geq 3Sr$$

where S the surface and V the volume. The equality $R = 3r$ only if all the above inequalities are equalities, that is the tetrahedron must be regular. The proof for E^n is the same.
We suppose that the tangent points A_1, B_1, C_1, D_1 of the faces of the tetrahedron $ABCD$ with the inscribed sphere (O, r) are the centroids of the faces respectively and G the centroid of the tetrahedron $ABCD$ The line OG intersects the altitude AH at the point O_1. The triangles AO_1G and A_1OG are similar, therefore:

$$\frac{AO_1}{OA} = \frac{AG}{GA_1} = 3$$

hence $AO_1 = 3r$. That is $O_1 = O$. Also $R = 3r$ and accordly the previous proof the tetrahedron $ABCD$ must be regular. We now can say that we have proved that:
Therem 3
The minimum circumscribed tetrahedron to a sphere is the regular.

Theorem 4
The regular tetrahedron has the min. surface from all the tetrahedrons circumscibed to a given sphere.

The proof is based in the in theorem 3. The formula of the volume is $(ABCD) = \frac{1}{3}rS$ where S the surface of $ABCD$. The volume is min. for the regular, that is min.S for the regular tetrahedron.
In the sequel we will prove two useful propositions in order to study similar problems.

Proposition a

We assume that the tetrahedron $ABCD$ does not inglude the pericenter. Then we can find a tetrahedron $A'BCD$ with bigger volume, bigger surface and bigger inradius.We suppose that the plane BCD intrsects the segmant OA where O the pericenter. Let $A'BCD$ the symmetric of the cap $ABCD$ with regard to the plane BCD and A' the symmetric of the point A. The half- line AA' intersects the sphere to the point A". The tetrahedron A"BCD has bigger volume,surface and circuradius.

Proposition 2

The inradius of the regular tetrahedron is $r = \frac{R}{3}$, where (O, R) is the circumsphere. So, for the inscribed tetrahedron $ABCD$ with distances from the pericenter to the faces p_a, p_b, p_c, p_d, both the followin cases
$(1) p_a, p_b, p_c, p_d < \frac{R}{3}$
$(2) p_a, p_b, p_c, p_d > \frac{R}{3}$
are impossible. The proof is easy.

Theorem 5

The regular tetrahdron has the max.volume from all the tetrahedrons inscribed in the same sphere.

Proof

We suppose that the tetrahedron $ABCD$ is inscribed in the sphere (O, R). Let p the parallel plane tangent to the sphere at the point A'. We choose p so that A, A' be in the same part with regard p. Obviously volume$(A'BCD) \geq (ABCD)$, equality only for $A = A'$.
We procced now by contradiction. We suppose that a no regular tetrahedron $ABCD$ has the max.volume and $AB < AC$. The plane q through A parallel to BCD is no tangent to the sphere. To see this, we considere the plane ABC which intersects the sphere to the circle ABC. The intersection of the plane q with the circle ABC is a segment.

Theorem 6

The regular tetrahedron inscribed in a sphere (O, R) has the max.surface from all the inscribed tetrahedrons.

Proof

Let $P = ABCD$ an inscribed tetrahedron, $P_1 = A_1B_1C_1D_1$ the circumscribed tetrahedron with the faces respectively parallel to P and $P_0 = A_0B_0C_0D_0$ the regular tetrahedron circumscribed to the sphere (O, R). We denote by $S(Q)$ the surface of the tetrahedron Q. The tetrahedrons P and P_1 are similar with ratio $\frac{r}{R}$. That is because r is the inradius of P and R the inradius of P_1. Therefore we will have:

$$\frac{S(P)}{S(P_1)} = \frac{r^2}{R^2}$$

From the well known inequality of Euler and theorem 4 we have:

$$\frac{r}{R} \leq \frac{1}{3}$$

$$S(P_1) \leq S(P_0)$$

Hence, from the above follows

$$S(P) \leq \frac{1}{9} S(P_0)$$

So the conclusion follows. The inscribed tetrahdron with max surface is the regular.

2^{nd} **Proof**

In a triangle ABC holds

$$a^2 + b^2 + c^2 \geq 4E\sqrt{3} \tag{8}$$

where a, b, c the sides and E the area.

Indeed, we know from the elementary Geometry that

$$a^2 + b^2 + c^2 = 4E(cotA + cotB + cotC)$$

but

$$cotA + cotB + cotC \geq \sqrt{3}$$

so ,the inequality (8) holds. Equality for ABC equilateral.

In the tetrahedron $ABCD$ inscribed in the sphere (O, R), we denote:

The edges DA, DB, DC by a', b', c' and AB, BC, CA by c, a, b.
We use (8) for the four triangles faces of $ABCD$ and we take.

$$2(a^2+b^2+c^2+a'^2+b'^2+c'^2) \geq 4F\sqrt{3} \tag{9}$$

where $F = \sum_1^4 E_i$, and E_i the area of the faces.
Now we use the formula of Leibniz

$$m\Big[\sum_1^{n+1} m_i|PA_i|^2\Big] = m^2|PQ|^2 + \sum_{i>j}^{n+1} m_i m_j |A_iA_j|^2$$

where $A_i(m_i)$ points with masses m_i. The center of mass is $Q = \frac{\sum m_i A_i}{m}$
$m = \sum m_i$ and P is some point.
For $P = O$ (circumcenter) , $m_i = 1$, we take:

$$4^2R^2 \geq a^2+b^2+c^2+a'^2+b'^2+c'^2 \tag{10}$$

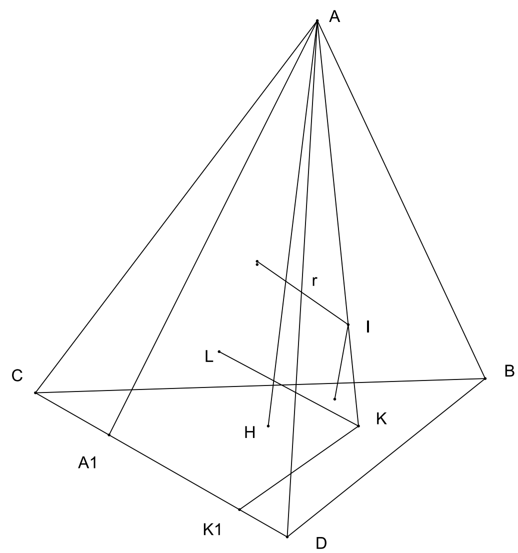
From (9)and (10) follows

$$8R^2 \geq F\sqrt{3}$$

That is:

$$max.F = \frac{8R^2}{\sqrt{3}}$$

hence, the equality for the regular tetrahedron, as follows from (8) and (10)
3^{rd} **Proof**
The bissectors planes of the dihedral angles AB, AC and AD passe through a line intersecting the plane ABC to the point K. The point K has equal distances from the planes of the trihedral angle $ABCD$.Let KL_1 the distance of K from the plane ACD, $AH = h_a$ the altitude from A, AA_1 the altitude from A to the triangle ACD and KK_1 the distance fom K the distance from K to CD



We have

$$(AKCD) = 1/3(ACD).KL_1 \qquad (AKCD) = 1/3.h_a.(KCD)$$

From the above, follows:

$$\frac{KL_1}{h_a} = \frac{(KCD)}{(ACD)}$$

As above we drow KK_2, KK_3 perpendiculars to BD, BC and AA_2, AA_3 perpendiculars to BD, BC. It no difficult to see that

$$\frac{AA_1}{KK_1} = \frac{AA_2}{KK_2} = \frac{AA_3}{KK_3} = p$$

and

$$\frac{(ACD)}{(KDC)} = \frac{ABD)}{(KBD)} = \frac{(ACB)}{(BCD)} = \frac{(ACD) + (ABD) + (ACB)}{(BCD)} = p$$

Setting

$$S = (ACD) + ABD) + (ABC)$$

we have

$$S = p(BCD)$$

Also

$$(KACD) = 1/3KL_1.(ACD), \quad (AKCD) = 1/3AH.(KCD)$$

therefore

$$\frac{AH}{KL_1} = \frac{(ACD)}{(KCD)} = \frac{CD.AA_1}{CD.KK_1} = \frac{AA_1}{KK_1} = p$$

The incenter the $ABCD$ is the point I in the segment AK . We denote by l_1, l_2, l_3, l_4 the barycentric coordinates of I. We have $l_i \geq 0$, $\sum l_i = 1$. also

$$\frac{r}{h_a} = l_1 \frac{1}{p} = \frac{KL_1}{AH} = \frac{l_1}{r}.KL_1$$

$$\frac{r}{KL_1} = pl_1$$

$$\frac{r}{KL_1} = 1 - l$$

that is

$$p = \frac{l_2 + l_3 + l_4}{l_1}$$

We considere the rerpendicular OM from the pericenter O to the BCD. The line MO intersects the circumsphere to the point M . We can suppose tho point O in the interior of the $ABCD$ and $OM \geq \frac{R}{3}$, we have:

$$\frac{AI}{IK} = \frac{l_2 + l_3 + l_4}{l_1} = \frac{h_a - r}{r} = \frac{h_a}{r} - 1 \leq \frac{MM'}{r} - 1$$

That is

$$\frac{AI}{IK} \leq \frac{R + OM}{OM} - 1 = \frac{R}{OM} + 1 - 1 = \frac{R}{OM} \leq 3$$

Hence

$$S \leq 3(BCD)$$

Thus for BCD=constand the max$S$ is $3(BCD)$. That is for $A \Rightarrow M'$ p is increasing. The Max when $A = M'$. The Max of the surface of $ABCD$ when BCD equilateral, that is $ABCD$ regular

The last part of this paper is a theorem about the sum of the edges of the tetrahedron $ABCD$.

Theorem 7

The regular tetrahedron has the max. of the sum of edges of the tetrahedrons inscribed in a sphere (O, R)

Proof

Let $ABCD$ be an iscribed tetrahedron in (O, R) and a, b, c, a', b', c' the edges (see the second proof of the thorem 5).

From (10) follows that

$$4\sqrt{6}.R \geq \sqrt{6}\sqrt{a^2+b^2+c^2+a'^2+b'^2+c'^2}$$

hence

$$6a_0 \geq a+b+c+a'+b'+c'$$

where a_0 the edge of the regular tetrahedron.

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