# Affine Geometry.The distance from a point to the line.

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**Trilinear coordinates**

Let $ABC$ be a triangle and a point $M$ in the plane,

$$MA' = m_a, MB' = m_b, MC' = m_c$$

the distances of the point $M$ from the sides respectively.We denote by $h_a, h_b, h_c$ the altitudes and

$$l_a = \frac{m_a}{h_a}, l_b = \frac{m_b}{h_b}, l_c = \frac{m_c}{h_c}$$

the barycentic coordinates of the point $M$.
In fact we introduce two kind of trillinear coordinates: The distances $MA' = m_a, MB' = m_b, MC' = m_c$ and the barycentric $l_a, l_b, l_c$. The problem is to express the distance of a point $M$ in trilinear coordinates from a line with coefficients in trilinear form.
We suppose now the orthogonal Cartesian system $XOY$ and $ABC$ a given triangle of reference. $OP$ the distance of the orizin from the site $BC$. We denote $OP = p_1$. We see that:
$m_a$ is the distance from $O$ from $BC$- the distance from $O$ from $e$, where $e$ is the paralelle to $BC$.
tha is:

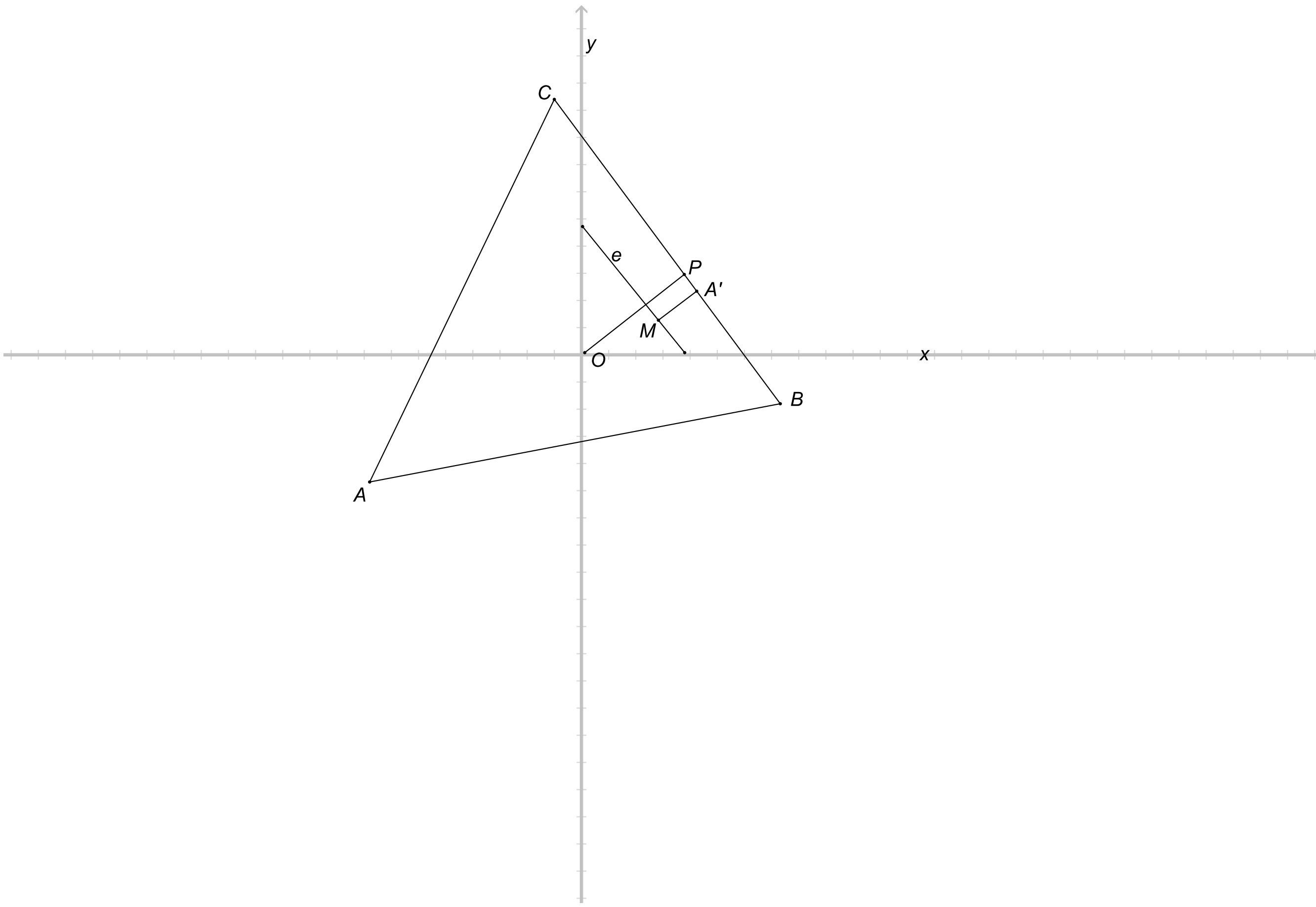
$$m_a = p_1 - (x_M cosu_1 - y_M sinu_1) \tag{1}$$

where $u_1$ is the angle of $OX, BC$. Similarly, we will take:

$$m_b = p_2 - (x_M cosu_2 - y_M sinu_2) \tag{2}$$

$$m_c = p_3 - (x_M cosu_3 - y_M sinu_3) \tag{3}$$

By $x_M, y_M$ the Cartesin coortinates of the point $M$ and $u_2, u_3$ the angles of $CA, AB$ with $OX$. See the fig.




Let be the line$L$ : $f(t) = q_1 t_a + q_2 t_b + q_3 t_c = 0$ where $t = (t_a, t_b, t_c)$ is the trilinear coordinates of the point $t$.
For some point $M'(m'_a, m'_b m'_c)$ in trilinear coordinates and $M(x, y)$ in Cartesian coordinates. We introduce the Cartesian coordinates in $L$.

$$f(M') = \sum q_1 \Big[ p'_1 - (xcosu_1 + ysinu_1 \Big]$$

. In Cartesian form is:

$$f(M') = x(q_1 cosu_1 + q_2 cosu_2 + q_3 cosu_3) + y(q_1 sinu_1 + q_2 sinu_2 + q_3 sinu_3)$$

$$-q_1 p_1 - +q_2 p_2 - +q_3 p'_3$$

So we will take

$$d(M,L) = \frac{x_M.A + y_M.B - q_1p_1 - q_2p_2 - q_3p_3}{\sqrt{A^2+B^2}}$$

where

$$A = q_1cosu_1 + q_2cosu_2 + q_3cosu_3$$

$$B = q_1sinu_1 + q_2sinu_2 + q_3sinu_3$$

and finaly

$$d(M,L) = \frac{q_1m_a + q_2m_b + q_3m_3}{\sqrt{q_1^2+q_2^2+q_3^2-2q_2q_3cosA-2q_3q_1cosB-2q_1q_2cosC}} \quad (4)$$

**Problem**
The point $A', B', C'$ are in the sides $BC, CA, AB$ of the triangle $ABC$ so that:

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B} = p$$

We suppose that the triangles $ABC$ and $A'B'C'$ have the same incenter. We will prove that the triangles are equilateral.
**Proof**
We will work in a trilinear system. Let $A(1,0,0), B(1,0,1), C(1,1,0)$, we also have:

$$\frac{CB'}{B'A} = \frac{B'-C}{A-B'} = p$$

$B'(1+p) = p.A+C = p(1,0,0)+(0,0,1) = (p,0,0)+(0,0,1) \Rightarrow B' = (k,0,1-k)$ where $k = \frac{p}{1+p}$. We also find $C'(1-k,k,0), A'(0,1-k,k)$.
The equations of the lines:

$$A'B' : x(1-k)^2 + yk^2 - zk(1-k) = 0 \quad (a)$$

$$B'C' : -xk(1-k) + y(1-k)^2 + zk^2 = 0 \quad (b)$$

$$C'A' : xk^2 - yk(1-k) + z(1-k)^2 = 0 \quad (c)$$

the trilinear distances of the incenter I with regard to the triangle $ABC$ (reference to $ABC$) are $I(r,r,r)$ where $r$ the inradius of $ABC$. From the

formula (4) we take $d(I, A'B'), d(I, B'C')$
Hence $d(I, A'B') = d(I, B'C') \Rightarrow$

$$(1-k)^4+k^4+k^2(1-k)^2-2k^2(1-k)^2cosC+2k(1-k)^3cosB+2k^3(1-k)cosA =$$

$$= (1-k)^4+k^4+(1-k)^2-2k^2(1-k)^2cosA+k^3(1-k)cosB+2k(1-k)^3cosC$$

After the manipulations for $k \neq 0, k \neq 1$ follows:

$$k = \frac{cosB - cosC}{2cosB - cosC - cosA}$$

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$$k = \frac{cosC - cosA}{2cosC - cosA - cosB}$$

after the manipulations we take:

$$cos^2A + cos^2B + cos^2C = cosAcosB + cosBcosC + cosCcosA$$

and finally $cosA = cosB = cosC$ that is $A = B = C$