An application of Brouwer's fixed point property theorem.

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1. The fixed point property is a topological invariant, that is if p is a continuous transformation to a set Q into itself and has the fixed point property, that is:
if

$$p(Q) \subset Q \quad \Rightarrow \exists x \in Q : \quad p(x) = x$$

then for every topological equivalent W of Q, the set W, has the fixed point property as well.
To see that, let h be a continuous transformation of W into itself: $h(W) \subset W$, and suppose u be a continuous transformation, so that: $u : \ Q \rightarrow W$.
The transformation $g = u^{-1}hu$ is continuous and transforms Q into Q, therefore, it has a double point y, that is, $u^{-1}(h(u(y))) = y$, or $h(u(y)) = u(y)$.
Hence, W has the fixed point property.

2. Let a point-set $A \ \in E^n$, $n \geq 3$, topological equivalent to the sphere $S^{(n-1)}$, where: $S^{(n-1)} = \{x/|x| = 1\}$.
We consider two decompositions of A into A_1, A_2 and A_1', A_2'. that is:

$$A = A_1 \cup A_2, \ \ and \ \ A = A_1' \cup A_2',$$

where, $A_1, \ A_2, \ A_1', \ A_2'$ are topological equivalent to $B^{(n-1)}$ ball: $B^{(n-1)} = \{x/|x| \leq 1\}$.

Suppose now that f is a continuous transformation of A into itself, $f(A) \subset A$, without fixed points, that is:

$$\forall x \in A : \ f(x) \neq x, \ \ and \ \ c_1 = A_1 \cap A_2 \quad and \ \ c_2 = A_1' \cap A_2'$$

then

$$c_1 \cap c_2 \neq 0$$

Proof
Let $A - A_1 \cap A_2 = X \cup Y$. The sets $X \cup (A_1 \cap A_2)$ and $Y \cup (A_1 \cap A_2)$ are topological equivalent to the $B^{(n-1)}$ ball.There are two possibilities: (a) there is at least one point $a \in X$,so that

$$f(a) \in Y \tag{1}$$

and (b)

$$f(X \cup A_1 \cap A_2) \subset X \cup A_1 \cap A_2. \tag{2}$$

But according Brouwer's theorem, from (2) the set $X \cup A_1 \cap A_2$ would have the fixed point property, contrary to our supposition.
Therefore (2) is no valid.
The set Y is connected and from (1) we conclude that:

$$f(X) \subset Y \cup A_1 \cap A_2 \;\; \textit{and similarly} \;\; f(Y) \subset X \cup A_1 \cap A_2 \tag{3}$$

Let now $c_1 \cap c_2 = 0$, then either $c_2 \subset X$ or $c_2 \subset Y$. That contradicts to (3) therefore $c_1 \cap c_2 \neq 0$.