A characterization of Euclidean space

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In this short note we will prove the following Geometrical characterization, for a Minkowski M^2 space (two dimensional normed space) to be Euclidean.

Theorem

If the tangent Minkowski segments from a point A to a Minkowski circle V are equal, then the Minkowski plane is an Euclidean plane, that is: $M^2 = E^2$.

Proof

Let V be a Minkowski circle in M^2 and from a point A the two tangent segments AB andAC to V. We suppose that: $\| AB \| = \| AC \|$.

In M^2 space a unit circle V_0 is centrosymmetric convex set and the norm of a line segment PQ is defined by:

$$\| PQ \| = \frac{2|PQ|}{|AB|}$$

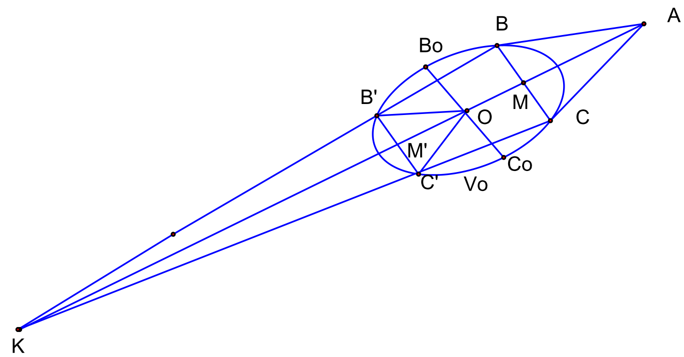
where $|PQ|$ is the Euclidean distance of the points P, Q and $|AB|$ the Euclidean distance of the parallel diameter of V_0.

The group of isometries in M^2 includes the product of homotheties and translations, so we can prove the theorem for the unit circle V_0.

Let V_0 be the unit circle in M^2 and from a point we drop the AB and AC tangent lines to V_0. We suppose that

$$\| AB \| = \| AC \| \ .$$

We consider the parallel radius OB' of V_0 to AB. the same AC' parallel to AC.



So we will have.

$$\parallel AB \parallel = \frac{|AC|}{|OB'|}, \quad \parallel AC \parallel = \frac{|AC|}{|OC'|}$$

see Fig.
So it follows that

$$\frac{|AC|}{|OB'|} = \frac{|AC|}{|OC'|}$$

That is the triangles BAC, $B'OC'$ are similar and hothetic and the center of homothety is the common point of the str.lines BB', AO, CC'. From the above we see that the ratio

$$\frac{BM}{MC} = \frac{B'M'}{M'C'}$$

so this ratio remains constant independent of the position of the point A on OK. We consider the point M' constant so the point A is moving in $M'O$ but BC remains parallel to $B'C'$ and the ratio $\frac{BM}{MC}$ is constant. Let B_0C_0 the parallel to $B'C'$ diameter of V_0. When BC is close to B_0C_0, that is $BC \to B_0C_0$, we will have:

$$\frac{BM}{CM} \to \frac{B_0O}{C_0O} = 1$$

so we conclude that

$$\frac{BM}{CM} = 1$$

Now according to the Bertrand theorem, see (1), we conclude that V_0 is an ellipse. The basic transformation group of M_2 is the affine group, so we can transform V_0 to a circle.

Cmment

From the above theorem we conglude that, if a special triangle is equilateral, then the Mincowski space is Euclidean.

References

1.A.C.Thompson, Minkowski Geometry,Cambridge Univ.Press.
2.Mahlon M. Day, Normed Linear Spaces, Springer-Verlag.