Relations between the distances of points in n-space

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It is our purpose here to establish some relations between the distances of points in R^n. Especialy we have proved some formulas working on the Darboux-Frobenius and Galey-Menger's determinants. Some of them have been represented in a more generalised form using instead of the points A_i the spheres (A_i, R_i).
From the well known part of the elementary Geometry there is a generalization of the Ptolemy's theorem in its plane version.
Let $F_i = (O, R_i)$,$i = 1, 2, 3, 4$ is a net of circles, then $A = 0$ where

$$A = \begin{vmatrix} -2R_1^2 & D(F_1,F_2) & D(F_1,F_3) & D(F_1,F_4) \\ D(F_2,F_1) & -2R_2^2 & D(F_2,F_3) & D(F_2,F_4) \\ D(F_3,F_1) & D(F_3,F_2) & -2R_3^2 & D(F_3,F_4) \\ D(F_4,F_1) & DF_4,F_2) & D(F_4,F_3) & -2R_4^2 \end{vmatrix}$$

$D(F_i, F_j)$ is the mutual power of the circles F_i, F_j.that is:

$$D(F_i, F_j) = O_iO_j^2 - R_i^2 - R_j^2$$

The converse holds as well.
If we took $R_i = 0$, $i = 1, 2, 3, 4$, $A = 0$ it is the well known Ptolemy's theorem.

1.The net of a family of spheres.
A family of spheres consist a net if there exist a point with the same power relative to every sphere.
Let

$$F_j = \sum_{i=1}^{n}(x_i - x_{ji})^2 - R_j^2 = 0, j = 1, 2, ...k \tag{1}$$

be a family of spheres in R^n.
We consider the linear system

$$y_1 x_{ji} + y_2 x_{j2} + ... + y_n x_{jn} + y_{n+1} + y_{n+2}\Big(\sum_{i=1}^{n}(x_{ji}^2 - R_j^2)\Big) = 0 \qquad (2)$$

and the matrix

$$A = \Big[\sum_{i=1}^{n}(x_{ji}^2 - R_j^2) \quad x_{j1} \quad x_{j2}.....x_{jn} \quad 1\Big]$$

of its coefficents.
Assuming that the system (2) is non trivialy compatible we must have

$$r < n + 2$$

where r is the rank of the matrix A.
(a) If $r = n + 1$, the system (2) has one solution $a_i = \frac{y_i}{y_{n+2}}, \quad i = 1, 2, ...n + 1$
Therefore we can write:

$$\sum_{i=1}^{n} x_{ji}^2 + 2\sum_{i=1}^{n}\frac{a_i}{2}x_{ji} + \sum_{i=1}^{n}\frac{a_i^2}{4} + \Big[-\sum_{i=1}^{n}\frac{a_i^2}{4} + a_{n+1}\Big] - R_j^2 = 0$$

for $j = 1, 2, ...k$
Hence:

$$\sum_{i=1}^{n}\Big(\frac{a_i}{2} + x_{ji}\Big)^2 - R_j^2 = \sum_{i}^{n}\frac{a_i^2}{4} - a_{n+1}$$

for $j = 1, 2, ...k$.
The left side of the above equality is the power of the point $A(a_1/2, a_2/2, ...a_n/n)$ relative to the spheres F_j for $j = 1, 2, ...k$ that is:

$$D(F_j, A) = constand \qquad (3)$$

so the spheres F_j consist a net of center the point A.
(b).For $r < n + 1$ the spheres have a radical axes a plane of order $n + 1 - r$.

2. The Darbou-Frobenius matrix

We denote

$$F_m = \sum_{i=1}^{n}(x_i - x'_{mi})^2 - R'^2_m = 0 \qquad (4)$$

$m = 1, 2, ...p$, one other sphere family, with matrix

$$A' = \begin{bmatrix} \sum_{i=1}^{n} x'^2_{mi} & - R'^2_m & x'_{m1} & x'_{m2}.. & x'_{mn} & 1 \end{bmatrix}$$

Also denoting

$$A_1 = \begin{bmatrix} 1 & -2x'_{m1} & -2x'_{m2}... & -2x'_{mn} & \sum_{i=1}^{n} x'^2_{mi} - R'^2_i \end{bmatrix}$$

we can easily see that:

$$A \cdot \tilde{A}_1 = \begin{bmatrix} D(F_1, F'_1) & D(F_1, F'_2) & \cdots & D(F_1, F'_p) \\ D(F_2, F'_1) & D(F_2, F'_2) & \cdots & D(F_2, F'_p) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D(F_k, F'_1) & D(F_k, F'_2) & \cdots & D(F_k, F'_p) \end{bmatrix} \quad (5)$$

where $D(F_j, F'_m)$ is the mutual power of the spheres F_j and F'_m. That is:

$$D(F_j, F'_m) = \sum_{i=1}^{n} (x_{ji} - x'_{mi})^2 - R_j^2 - R'^2_m$$

The matrix $A \cdot \tilde{A}_1$ will be denoted by

$$D(F_1, F_2, ...F_k : F'_1, F'_2, ...F'_p) = A \cdot \tilde{A}_1$$

and we call it the Darboux-Frobenius matrix. We shall study the Darboux-Frobenious determinant.

(a). Let $k, p > n + 2$.

The matrices A, A' are of the maximum rank $n + 2$,so every minor of the determinant $D(F_1, F_2, ...F_k : F'_1, F'_2, ...F'_p)$ with rank greater than $n + 2$ will be zero.

(b) if $min(k, p) = n + 2$ and one at least from $detA, detA'$ has rank $n + 1$, then every minor of the $detD(F_1, F_2, ..F_k : F'_1, F'_2, ...F'_p)$ with rank $n + 2$ will be zero.

(c) if $min(k, p) \leq n + 1$ and r is the rank of one of the A, A', then every minor from the determinant $D(F_1, F_2, ..F_k : F'_1, F'_2, ...F'_p)$ with rank less than r will be zero.

Taking now $R'_m = 0 \quad m = 1, 2, ...p$ the spheres F'_m will be the points

$A'_1, A'_2, ...A'_p$ (their centers).
Here the $D(F_j, A'_m)$ must be the power of the point A'_m with respect the sphere F_j. The relations (a),(b),(c) are transformed analogously.
Besides if we take $R_j = 0$, we have: $D(A_j, A_m) = |A_j\vec{A}_m|^2$, that is the distance of the points A_j, A_m. The matrix $D(A_1, A_2, ..A_K : A'_1, A'_2...A'_p)$ gives for the cases a),b),c) analogous relations.

3. **Generalization of the Ptolemy's theorem**
The Darboux-Frobenious determinant can be written

$$P = detD(F_1, F_2, ...F_{n+2} : F_1, F_2, ...F_{n+2}) = (-1)^n 2^n (detA)^2 \qquad (6)$$

for a family of spheres in R^n. Assuming that the above spheres consist a net, then $P = 0$. The converse is obvious, therefore the following theorem holds.
Theorem
Iff for the family of the spheres $F_1, F_2, ...F_{n+2}$ holds $P = detD(F_1, F_2, ...F_{n+2} : F_1, F_2, ...F_{n+2}) = 0$ then the spheres consist a net.
For R^2, F_1, F_2, F_3, F_4 are a net of circles in the plane. That is, there is a point, the center, with the same power relative to the circles. So the tangents from the center to the circles would be equal. If the radii of the circles are zero, the centers will be in the circle with center the center of the net.
Some restrictions on the above theorem can lead us to Ptolemy's theorem. Indeed, we assume that $R_i = 0$ for $i = 1, 2, ...n + 2$ and $n = 2$. Then we must have:

$$P = detD(A_1, ...A_4 : A_1, ...A_4) = \begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix} = (-1)^2 2^2 (detA)^2$$

where

$$detA = \begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix}$$

$A_i = (x_i, y_i)$ and $|A_i\vec{A}_j|^2 = d_{ij}^2 = d_{ji}^2$. It is no difficult to find out that:

$$P = -(d_{12}d_{34}+d_{13}d_{24}+d_{14}d_{23})(-d_{12}d_{34}+d_{13}d_{24}+d_{14}d_{23})(d_{12}d_{34}-d_{13}d_{24}+d_{14}d_{23})$$
$$(d_{12}d_{34} + d_{13}d_{24} - d_{14}d_{23})$$

and Ptolemy's theorem follows.
Also from the formula (6) arises:

$$detD(F_1, F_2, ...F_{n+2} : F_1, F_2, ...F_{n+2}) \geq 0 \quad for \quad n \quad even \quad and \leq 0 \quad for \quad n \quad odd \tag{7}$$

For the points $A_1, A_2, ...A_{n+2}$ in R^n.

$$detD(A_1, A_2, ...A_{n+2} : A_1, A_2, ...A_{n+2}) \geq 0 \quad for \quad n \quad even \quad and \leq 0 \quad for \quad n \quad odd \tag{8}$$

An interesting formula is the following

$$detD(A_1..A_{n+2} : A'_1..A'_{n+2}) = detD(A_1..A_{n+2} : A_1..A_{n+2}) \cdot detD(A'_1A'_2..A'_{n+2} : A'_1A'_2..A'_{n+2}) \tag{9}$$

Where A_i and A'_i, $i = 1, 2, ...n + 2$ are points in R^n.
The proof follows from the formula (5). We have

$$detD(A_1A_2..A_{n+2} : A'_1A'_2..A'_{n+2}) = det(A \cdot \tilde{A}_1) = (-1)^{n+1}2^n detA \cdot detA'$$

and

$$detD(A_1A_2..A_{n+2} : A_1A_2..A_{n+2}) = (-1)^{n+1}2^n(detA)^2$$
$$detD(A'_1A'_2..A'_{n+2} : A'_1A'_2..A'_{n+2}) = (-1)^{n+1}2^n(detA')^2$$

From the above follows (9).

5. The Cayley-Menger Matrix.

It is so kaled the matrix

$$\Theta(A_1A_2..A_r : A'_1A'_2..A'_k) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \vec{A_1A'_1}^2 & \cdots & \vec{A_1A'_k}^2 \\ . & & & \\ 1 & \vec{A_rA'_1}^2 & \ldots & \vec{A_rA'_k}^2 \end{bmatrix}$$

where $A_1, A_2, ..A_r$ and $A'_1, A'_2, ..A'_r$ two point sets in R^n.

Proposition 1.

For a point $O \in R^n$ and the two point sets $A_0, A_1, ..A_r, \quad A'_0, A'_1, ..A'_r$ holds:

$$(a) \quad det\Theta(A_0, ..A_r : A'_0, ..A'_r) = (-2)^r \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & \vec{OA_0} \cdot \vec{OA'_0} & \cdots & \vec{OA_0} \cdot \vec{OA'_r} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \vec{OA_r} \cdot \vec{OA'_r} \end{vmatrix}$$

(b) We denote $\vec{A_0A_i} = \vec{x_i}$ and $\vec{A_0'A_i'} = \vec{y_i}$ for $i = 1, 2, ...r$ and $G(x, y)$ the Gramian of the $\vec{x_i}, \vec{y_i}$ that is:

$$G(x,y) = \left[\begin{array}{cccc} \vec{x_1}\vec{y_1} & \vec{x_1}\vec{y_2} & \cdots & \vec{x_1}\vec{y_r} \\ \vec{x_2}\vec{y_1} & \vec{x_2}\vec{y_2} & \cdots & \vec{x_2}\vec{y_r} \\ \cdots & \cdots & \cdots & \cdots \\ \vec{x_r}\vec{y_1} & \vec{x_r}\vec{y_2} & \cdots & \vec{x_r}\vec{y_r} \end{array} \right|$$

we slall prove

$$det\Theta(A_0, A_1, ..A_r : A_0'A_1', ..A_r') = (-1)^{r+1}2^r detG(x, y)$$

Proof

(a). For every point $O \in R^n$ we have:

$$|\vec{A_iA_j}|^2 = |\vec{OA_i}|^2 + |\vec{OA_j}|^2 - 2\vec{OA_i} \cdot \vec{QA_j}$$

We substitute from the above formula $|\vec{A_iA_j}|^2$ for $i, j = 1, 2, ...r$ in $det\Theta(A_0, ..A_r : A_0', ..A_r')$. We multiplay the first row by $|\vec{OA_i}|^2$ and substract it from the $i+2$ row. Then we multiplay the first column by $|\vec{OA_j}|^2$ and we substract it from the $j + 2$ column. Then an easy calculation gives (a).

(b). We take $det\Theta(A_0..A_r : A_O'..A_r')$ and we substract the second row from its following ones. then we substract the first column from the others.

So we will have

$$det\Theta(A_0A_1, ..A_r : A_0'A_1', ..A_r') = (-1)^{r+1}2^r detG(x, y)$$

Proposition 2.

If the points $A_0, A_1, ..A_{k+1}$ lie in a $q - plane$ where $q = n - k$, then for every point set $A_0', A_1', ...A_{k+1}' \in R^n$ we will have:

$$det\Theta(A_0A_1...A_{k+1} : A_0'A_1'...A_{k+1}') = 0$$

If the two point sets $A_0, A_1, ...A_k$ and $A_0', A_1', ...A_k'$ define two different orthogonal $(n - k - 1)$-planes then:

$$det\Theta(A_0A_1...A_k : A_0'A_1'...A_k') = 0$$

Proof.

For the liearly dependent point set $A_0, A_1, ...A_{k+1}$, there exist the real numbers $p_0, p_1, ..p_{k+1}$ (non all zero) so that, for every point $O \in R^n$ we shaal have:

$$\sum_{i=0}^{k+1} p_i = 0 \bigwedge \sum_{i=0}^{k+1} p_i\vec{OA_i} = 0$$

or,equivalently

$$\sum_{i=1}^{k+1} p_i \vec{A_0 A_i} = 0 \quad or \quad \sum_{i=1}^{k+1} p_i \vec{A_0 A_i} \cdot \vec{A'_0 A'_j} = 0$$

for $j = 1, 2, ..k + 1$
But the above system has no trivial solution. So the determinant of the coefficients must be zero. Thus if we denote $\vec{A_0 A_i} = \vec{x_i}, \quad \vec{A'_0 A'_j} = \vec{y_j}$ we will have:

$$detG(x, y) = 0$$

The above and (5) proposition1(b) prove the asked.
(b). Let L be the linear subspace spanned by the vectors $\vec{A_0 A_i} = \vec{x_i}$ and L' the linear space spanned by the vectors $\vec{A'_0 A'_j} = \vec{y_j}$ for $i, j = 1, 2, .., k$. According the above we can take a vector $\vec{t} \neq 0$ of L so that

$$\vec{t} \cdot \vec{y_i} = 0$$

for $i = 1, 2, ...k$.
Let it be $\vec{t} = \sum_{i=1}^{k} q_i \vec{x_i}$ with $q_i \in R, \quad i = 1, 2, ...k$
So, we will have

$$\sum_{i=1}^{k} q_i \vec{x_i} \cdot \vec{y_j} = 0$$

The above system has no trivial solution. That is

$$detG(x, y) = 0$$

therefore

$$det\Theta(A_0 A_1 ... A_k : A'_0 A'_1 ... A'_k) = 0$$

Proposition 3. If

$$det\Theta(A_0, A_1 .. A_k : A'_0, A'_1, ..A'_k) = 0$$

then
(a) One at least from the point sets $A_0, A_1, ..A_k$ and $A'_0, A'_1, ...A'_k$ contains linearly depentent points, or
(b) The point sets $A_0, A_1, ..A_k$ and $A'_0, A'_1, ..A'_k$ belong in two orthogonal linear subspaces of R^n.
Proof.

From 5 prop. 1(b) follows
$det\Theta(A_0...A_k : A'_0...A'_k) = 0$ therefore $detG(x,y) = 0$
where $\vec{A_0A_i} = \vec{x_i}, \quad \vec{A'_0A_j} = \vec{y_j}$
but, then, the system

$$y_j(\sum_{i=1}^{k} p_i\vec{x_i}) = 0, \qquad j = 1, 2, ...k$$

has no trivial solution. That is
(a)

$$\sum_{i=1}^{k} p_i\vec{x_i} = 0$$

or if

$$\sum_{i=1}^{k} p_i\vec{x_i} \neq 0$$

then

$\sum_{i=1}^{k} p_i\vec{x_i}$ is orthogonal to $\vec{y_j}$.

Proposition 4.
If $A_0, A_1, ..A_k$ and $A'_0, A'_1, ..A'_k$ are two sets in R^n, then:

$$det\Theta(A_0..A_k : A'_0..A'_k)^2 \leq det\Theta(A_0..A_k : A_0..A_k)det\Theta(A'_O..A'_k : A_0..A'_k) \tag{10}$$

The proof follows immediately from Bessel-Schwarz inequality for Gramians, see (4) and from 5 proposition 1(b).
The equality follows from 5 prop. 1(b) and from (4). It holds
(a). If the two linear spaces L, L' which are spanned by the point sets, are parallel or coincided and conversaly.
(b). If at least one from the point sets contains linearly dependent points and conversaly.
It is well known that the volume $V(A_0, A_1, ..A_k)$ of a simplex with vertices $A_0, A_1, ..A_k$ is given by the formula

$$(-1)^{k+1}2^k(k!)^2[V(A_0A_1, ..A_k)]^2 = det\Theta(A_0A_1..A_k : A_0A_2..A_k)$$

se (2)From the above and formula 10 we take

$$|det\Theta(A_0A_1..A_k : A'_0A'_1..A'_k)| \leq 2^k(k!)^2|V(A_0A_1..A_k)V(A'_0A'_1..A'_k)| \tag{11}$$

The equality occurs iff (a) and (b) holds.
Remark.
Formula (11) can be used for a defination of the angle between the two linear spaces. So, if the two point sets $A_9, A_1, ..A_k$ and $A'_0, A'_1..A'_k$ spann respectively the linear spaces L and L', then the angle ϕ between them can be defined by

$$cos\phi = \frac{det\Theta(A_0A_1..A_k : A'_0A'_1..A'_k)}{2^k(k!)^2V(A_0A_1..A_k)V(A'_0A'_1..A'_k)}$$

Relations between Darboux-Frobenius and Cayley-Menger's determinants Proposition 1.
Let $s_k = A_0A_1..A_k$ be a k-simplex and R its circumradius. We will prove that

$$detD(A_0A_1..A_k : A_0A_1..A_k) + 2R^2det\Theta(A_0A_1..A_k : A_0A_1..A_k) = 0$$

Proof
Let O be the circumcenter of the s_k. From (5) prop. 1(b) follows

$$det\Theta(A_0, A_1, ..A_k, O : A_0, A_1, ..A_k, O) = 0$$

so, we have:

$$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & R^2 \\ \multicolumn{4}{c}{\dots\dots\dots\dots} \\ 1 & R^2 & \cdots & 0 \end{vmatrix} = 0$$

We multiplay the first row by R^2 and we substract it from the last row. Then we multiplay the first column by R^2 and we substract it from the last column. Expanting the determinant with respect to the elements of the last column we have the proposed, see(5).
Proposition 2.
We denote by F and F' The circumspheres of the k-simplices
$s_k = A_0A_1..A_k$ and $A'_0A'_1..A'_k$ and $D(F, F')$ their mutual power, that is $D(F'F') = OO'^2 - R^2 - R'^2$ where O, O' the circumcenters. We will prove that:

$$detD(A_0A_1..A_k : A'_0A'_1..A'_k) + D(F, F')det\Theta(A_0A_1..A_k : A'_0A'_1..A'_k) = 0$$

Proof

Taking into account 5 prop.1(b), we will have:

$$det\Theta(A_o, A_1, ..A_k, O : A'_0, A'_1, ..A'_k, O) = 0$$

hence,

$$\begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & \vec{A_0A'_0}^2 & \dots & R^2 \\ \dots & \dots & \dots & \dots \\ 1 & R'^2 & R'^2 \cdots & \vec{OO'}^2 \end{vmatrix} = 0$$

We multiplay the first row by R'^2 and we substract it from teh last row. Further, we multiplay the first column by R^2 and we substract it from the last column. By expansion of the determinant with respect to the elements of last column we have:

$$detD(A_0..A_k : A'_0..A'_k) + (OO'^2 - R^2 - R'^2)det\Theta(A_0..A_k : A'_0..A'_k) = 0$$

Concluding from (9) and (6) prop. 1 and 2 we take:

$$det\Theta(A_0..A_k : A_0..A_k)\Theta(A'_0..A'_k : A'_0..A'_k) = \left[\frac{D(F, F')}{2rr'}\right]^2 \left[det\Theta(A_0..A_k : A'_0..A'_l)\right]^2$$

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