Two elementary inequalities for the proof of the Brunn-Minkowski Theorem.

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R. Schneider in the proof of the Brunn-Minkowski theorem uses the following inequality.

$$\left[(1-\lambda)u_0^p + \lambda u_1^p\right]^{\frac{1}{p}}\left[\frac{1-\lambda}{u_0} + \frac{\lambda}{u_1}\right] \geq 1. \qquad (1)$$

Where $0 \leq \lambda \leq 1$ u_0, u_1 positive real numbers and p positive integral.
The elementary proof below is quite interesting.
We take

$$1-\lambda = \frac{\mu}{\mu+\nu}, \lambda = \frac{\nu}{\mu+\nu} u_0 = x, u_1 = y$$

and (1) is transformed to

$$\left(\mu x^p + \nu y^p\right)\left(\frac{\mu}{x} + \frac{\nu}{y}\right)^p \geq \left(\mu + nu\right)^{p+1} \qquad (2)$$

From A.M-G.M follows

$$\frac{\mu}{x} + \frac{\nu}{y} \geq (\mu+\nu)\left(\frac{1}{x}\right)^{\frac{\mu}{\mu+\nu}}\left(\frac{1}{y}\right)^{\frac{\nu}{\mu+\nu}}$$

or

$$\left(\frac{\mu}{x} + \frac{\nu}{y}\right)^p \geq (\mu+\nu)^p\left[\left(\frac{1}{x}\right)^{\frac{\mu}{\mu+\nu}}\left(\frac{1}{y}\right)^{\frac{\nu}{\mu+\nu}}\right]^p$$

Also from A.M-G.M we take

$$\frac{\mu x^p + \nu y^p}{\mu+\nu} \geq \left[x^{\frac{\mu}{\mu+\nu}} \cdot y^{\frac{\nu}{\mu+\nu}}\right]^p$$

From the last two inequalities we take (2).
P.Gruber for the same proof uses the inequality

$$\left(\frac{1}{v^{d-1}}+\frac{1}{w^{d-1}}\right)^{d-1}\left(\frac{V}{v}+\frac{W}{w}\right)\geq\left(V^{\frac{1}{d}}+W^{\frac{1}{d}}\right)^{d} \tag{3}$$

We put:

$$\mu=V^{\frac{1}{d}},\nu=W^{\frac{1}{d}},p+1=d,\mu x^{p}=\frac{V}{v}\Rightarrow\frac{\mu}{x}=v^{\frac{1}{d-1}},,\nu y^{p}=\frac{W}{w}\Rightarrow\frac{\nu}{y}=w^{\frac{1}{d-1}}$$

in (2) and we take (3).

References

1.R.Schneider, Convex bodies: The Brunn-Minkowski Theory.Cambridge university press.
2.P.M.Gruber, Convex and Discrete Geometry Draft.