Three problems on mixed area

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Problem 1. Betke-Weil
Let F_1, F_2 convex be sets in the plane, $V(F_1, F_2)$ is the mixed volumes and L_1, L_2 are the perimeters. Then it is:

$$V(F_1, F_2) \leq \frac{L_1 L_2}{8}$$

Proof

We suppose that the points A, B bissect the perimeter of F_2. O is the middle point of AB and $M \in bdF_1$ and l the tangent at the point M of F_2 and p_2 the support function. From the triangle AMB we see that

$$OM \leq \frac{MA + MB}{2}$$

So we have $p_2 \leq OM \leq \frac{L_2}{4}$, therefore:

$$V(F_1, F_2) = \frac{1}{2} \oint p_2 ds_1 \leq \frac{L_2}{8} \oint ds_1 = \frac{L_1 L_2}{8}$$

The equality if and only if F_1 and F_2 are perpendicular straight line segments.

Problem 2, Betke-Weis
For the convex set F holds:

$$V(F, -F) \leq \frac{\sqrt{3}}{18} L^2$$

Proof

The proof is based in the following problem of Eggleston.(Problems in Euc.space,Pergamon press,p 157(iii)).
If Γ is central then every circumscribing equilateral triangle has perimeter less or equal to $L\sqrt{3}$.
We consider $F^{\odot} = \frac{F-F}{2}$ and the min regular hexagon $Q = ABCDEF$ circuscribed to $F^{\odot}$ and as we know to both F and $-F$.
Taking the center of Q as orizin we will have:

$$V(F,-F) = \frac{1}{2}\oint p_F ds_{-F}$$

But $AB = a \geq p_F$, therefore

$$V(F,-F) \leq \frac{6a}{2.6}L$$

The intersections of the lines AB, CD, EF define the equilateral triangle KPM with side $3a$. So according to the above problem of Eggleston we will have:

$$V(F,-F) \leq \frac{L\sqrt{3}}{2.6}L.\frac{2}{3} = \frac{\sqrt{3}}{18}L^2.$$

As it is easily understood until now we considered F smooth. For F polygon we can determine the case of the equality. We first see that it must be regular and then with min. number of sides, that is a equilateral triangle. We did'n't find a case of equality for F smooth.

Problem 3. L.Sandalo
Mixed volumes and duality

$$V(F,F^*) \geq \pi$$

by F^* we denote the polar dual of F.
Proof

$$V(F,F^* = \frac{1}{2}\int_0^L p^*(s)ds \tag{1}$$

but $r(\phi)p^*(\phi) = 1$, also $p(\phi) \geq r(\phi)$ therefore

$$p^*(\phi)p(\phi) \geq \ 1 \tag{2}$$

From (1) and(2) follows:

$$V(F,F^*) \geq \int_0^{2\pi} \frac{ds(\phi)}{p(\phi)} = \frac{1}{2}\int_0^{2\pi} \frac{\rho(\phi)d\phi}{p(\phi)}$$

$$= \int_0^{2\pi} \frac{p+\ddot{p}}{p} d\phi = \pi + \frac{\dot{p}}{p}\Big/_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \dot{p} d(\frac{1}{p}) =$$

$$= \pi + \frac{1}{2}\int_0^{2\pi} \frac{\dot{p}^2}{p^2} \geq \pi$$

Equality for $p(\phi) = r(\phi)$ and $\dot{p} = 0$ that is F =circle.

References

1.R.Schneider, Convex bodies: The Brunn-Minkowski Theory. Cambridge university press.

L.A.Santalo,Integral Geometry and Geometrc Probability, Addison-Wesley Publishing Company.