# The cutting numbers.

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Let $F_0$ and $F$ be convex compact smooth figures in $E^2$. We see that we can find a point $z \in R^2$ and a number $t \in R^+$ so that: $z + tF \in F_0$.
We will call inradius of $F_0$ relative to $F$ the max $t$. We denote $r_F(F_0)$.
With the same way we can define the circumradius of the $F_0$ relative to $F$, that is the min $t$ so that $z + tF \ni F_0$ and we denote $R_F(F_0)$.
According the above, we conclude that for every number $t$ so that

$$r_F(F_0) < t < R_F(F_0)$$

and for every $z \in E^2$ the figure $z+tF$ is not included in $F_0$ and does't include $F_0$.
We will call **cutting number** of $F_0$ relative to $F$, this number $t$. We denote: $t_F(F_0)$, or simply $t$. That is:

$$(z + tF \notin intF_0) \bigwedge (int(z + tF) \not\ni F_0)$$

For the present paper we need two simple propositions on the cutting numbers.
**P.1**

$$t_F(F_0).t_{F_0}(F) = 1$$

The proof is very easy.
**P.2**
For $t = t_F(F_0)$ we can find $z$ so that the figures $F_0$ , $z + tF$ are inscribed in the same triangle $ABC$.
Indeet, we consider the insrcibed figure $F_1 = r_F(F_0).F$ to $F_0$. For $t$ between the numbers $r_F(F_0), R_F(F_0)$ we find an honothetic to $F_1$ figure $F'$ including $F_1$ and included in $R_F(F0).F$. The convex cover of $F_0 \cup F'$ has a straight line segments in the perimeter. So there is a triangle circumscribed to $F_0 \cup F'$

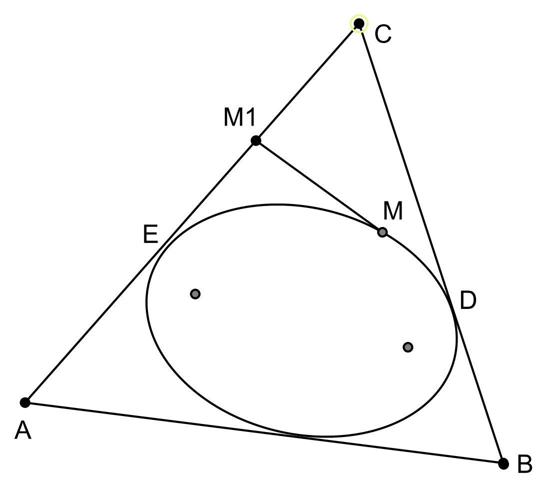
and to both $F_0$, $F'$.
Some interesting theorems can be proven using the cutting numbers.
**Theorem 1**
For the convex figures $F_0$ and $F' = tF$ where $t = t_F(F_0)$ holds:

$$(V(sF_0 + s'F') \geq sV(F_0) + s'V(F') \quad (1)$$

with $s, s' > 0$ and $s + s' = 1$.
that is for the set $\{F_0, F' + tF\}$, for $r_F(F_0) < t < R_F(F_0)$, the volume is linear (concave).
**Proof** of the theorem 1.
We translate $F'$ as in Proposition 2 exposed and let the triangle $ABC$ circumscribed to both figures $F_0$ and $F'$. (or we translate $F' = r_F(ABC).F$ inside in a circumscribed tringle to $F_0$).
Now we take a point M of the arc DE of $bdF_0$. $MM_1$ is the tangent segment until the side $AC$ and we use a formula of Frobenius. That is: The area of the part included between the arc DE and the angle $C$ is:

$$\frac{1}{2}\int_0^{\pi - C} MM_1^2 d\phi.$$

where $\phi$ is the angle of $MM_1$ with the side $AC$



The same for $F'$ and $M'M'_1$, the analogous segment. The figure $sF_0+s'F'$, $s, s' > 0$ $s+s' = 1$ is inscribed in the triangle $ABC$ and the analogous segment is $MM_1 + s'M'M'_1$. We put $MM_1 = p$ and $M'M'_1 = p'$.
Easily we find

$$V(F_0) = V(ABC) - \frac{1}{2}\int_0^{2\pi} p^2 d\phi \tag{2}$$

Therefore

$$P = V(sF_0 + s'F') = V(ABC) - \frac{1}{2}\int_0^{2\pi} (sp + s'p')^2 d\phi$$

From the well knwn inequality $sp^2 + s'p'^2 \geq (sp + s'p')^2$ follows

$$P \geq (s + s')V(ABC) - \frac{1}{2}\Big[\int_0^{2\pi} sp^2 d\phi + \int_0^{\pi} s'p'^2 d\phi\Big]$$

or

$$P \geq s\Big[V(ABC) - \frac{1}{2}\int_0^{2\pi} p^2 d\phi\Big] + s'\Big[V(ABC) - \int_0^{2\pi} p'^2 d\phi\Big]$$

finally

$$V(sF_0 + s'F') \geq sV(F_0) + s'V(F')$$

The case of the equality when $sp = s'p'$. Here from the elementaty Geometry we conclude that $F_0 = F'$, or $F_0 = tF$ that is $F_0$ and $F$ must be similar.
A number of inequalities can be obtained from our basic inequality (1). for $s = s' = \frac{1}{2}$ we take:

$$V(\frac{F_0 + F'}{2}) \geq \frac{V(F_0) + V(F')}{2} \tag{3}$$

It is known that:

$$V(\frac{F_0 + F'}{2}) = \frac{V(F_0)}{4} + \frac{V(F')}{4} + \frac{V(F_0, F')}{8}$$

where $V(F_0, F')$ the mixed area. The last inequality according to (3) gives:

$$V(F_0, F') \geq \frac{V(F_0) + V(F')}{2} \tag{4}$$

From (4) and the A.M-G.M inequality we take

$$V(F_0, F')^2 \geq V(F_0)V(F') \tag{5}$$

We substitude in (4) and (5) $F' = tF$. So we have

$$tV(F_0, F) \geq \frac{V(F_0) + t^2 V(F)}{2} \tag{6}$$

$$V(F_0, F)^2 \geq V(F_0)V(F) \tag{7}$$

Also from

$$V(sF_0 + s'F) = s^2 V(F_0) + 2ss'V(F_0, F) + s'^2 V(F)$$

and from (7), we take:

**Theorem 2**.

$$V(sF_0 + s'F)^{\frac{1}{2}} \geq sV(F_0)^{\frac{1}{2}} + s'V(F)^{\frac{1}{2}} \tag{8}$$

The Brunn-Minkowski Theorem.

**Isoperimetric inequalities**

We suppose that $F = u$ is the unit circle and $r_0$ the inradius of $F_0$,formula (6) will be.

$$r_0 V(F_0, u) \geq \frac{V(F_0) + r_0^2 V(u)}{2}$$

hence,

$$r_0 \frac{L_0}{2} \geq \frac{V(F_0) + \pi r_0^2}{2}$$

That is beaucause of the well known $V(F_0, u) = \frac{L_0}{2}$ .
The last relation can be written

$$\frac{L_0^2}{4\pi} - V(F_0) \geq \frac{1}{4\pi}(L_0 - 2\pi r_0)^2 \tag{9}$$

That is an isoperimetric inequality of Bonnesen style.
The inequality (6) can be written

$$0 \geq t^2 V(F) - 2tV(F_0, F) + V(F) \tag{10}$$

From (10) we see that the cutting numbers of $F_0$ relative to $F$ are included between the real roots of the equality (10), $\frac{V(F_0,F)+\sqrt{D}}{V(F)}, \frac{V(F_0,F)-\sqrt{D}}{V(F)}$ where

$D = V(F_0, F)^2 - V(F_0)V(F)$.
Let now $\rho_0 = r_F(F_0)$ and $P_0 = R_F(F_0)$. It is:

$$\frac{V(F_0, F) - \sqrt{D}}{VF)} \leq \rho_0 \leq P_0 \leq \frac{V(F_0, F) + \sqrt{D}}{VF)} \tag{11}$$

Therefore

$$P_0 - \rho_0 \leq \frac{2\sqrt{D}}{V(F)}$$

and finally

$$V(F_0, F)^2 - V(F_0)V(F) \geq \frac{VF)^2(P_0 - \rho_0)^2}{4} \tag{12}$$

Symmetrically we can write

$$V(F_0, F)^2 - V(F_0)V(F) \geq \frac{V(F_0)^2}{4}(\frac{1}{\rho_0} - \frac{1}{P_0})^2 \tag{13}$$

From (12) and (13) follows

$$V(F_0, F)^2 - V(F_0)V(F) \geq \frac{V(F_0)V(F)(P_0 - \rho_0)^2}{4\rho_0 R_0}$$

or

$$V(F_0, F)^2 \geq V(F)V(F_0)\frac{(R_0 + \rho_0)^2}{4R_0\rho_0} \tag{14}$$

From (12) we can take the very well known inequality of Bonnesen for $F = u$ the unit circle and $r_0, R_0$ the inradius and the circumradius of $F_0$.

$$L_0^2 - 4\pi V(F_0) \geq \pi^2(R_0 - r_0)^2 \tag{15}$$

A very interesting inequality follows from (7)

$$V(F_0, F)^2 - V(F_0)V(F) \geq \Big[\frac{V(_0) - t^2V(F)}{2t}\Big]^2 \tag{16}$$

where $t = t_F(F_0)$
Many other inequalities can be taken from (6) and (14) e.g. for $F' = \frac{L_0}{L}L$, that is for $t = \frac{L_0}{L}$

$$V(F_0, F)^2 - V(F_0)V(F) \geq \frac{\Big[L^2V(F_0) - L_0^2V(F)\Big]^2}{4L_0^2L^2} \tag{17}$$

Also for $t = \frac{L_0}{L}$ the inequality (5) will be

$$0 \geq \frac{V(F0)}{L_0^2} - 2\frac{V(F_0, F)}{L_0 L} + \frac{V(F)}{L^2} \tag{18}$$

The inequality of Frobenius.

**P.s.**

Another easy proof of Brunn-Mincowski theorem can be as bellow.
Let (l.l') and (q,q') two parallel strips of the convex figures $F_0$ and $F$ of direction $\vec{u}$ and $d_0$ and $d$ the breaths respectively. The homothetic $F' = tF$ where $t = \frac{d_0}{d}$ can be translated in the strip (l,l'). The line m of direction $\vec{u}$ intersects $F_0$ ,$F$ and $F_s = sF_0 + s'F'$, $s, s' > 0, s + s' = 1$ into the segments $p, p', p_s$. We will have

$$\int_a (sp + s'p')da \geq \int_a spda + \int_a s'p'da$$

where $a$ the distance between (l,l').
So we take:

$$V(sF_0 + s'F) \geq sV(F_0) + s'V(F')$$

Probably you have the question, why the first proof. That is because of the variationality of the cutting numbers.