A charasteric property of the ellipse

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Problem

Let (c) be a compact smooth convex curve in E^2 and A,B are interior points. The point $M \in (c)$ and we denote $\epsilon(M)$ the support line of the point M and $\vec{\epsilon_0}$ the unit tangent vector at the point M. We suppose that:

$$\angle(\vec{AM}, \vec{\epsilon_0}) = \angle(\vec{BM}, -\vec{\epsilon}_0) \tag{1}$$

Prove that (c) is an ellipse.
Formulate the analogous problem for the hyperbola and the parabola.
Proof
Let O an interior point and $\vec{r} = r(\vec{M})$ the position vector of the point $M \in (c)$.
Suppose $\phi = \angle(\vec{r}, \vec{\epsilon_0})$. We will have:
$\vec{r} = r\vec{r_0}$ or,

$$\frac{d\vec{r}}{ds} = \vec{\epsilon_0} = \frac{dr}{ds}\vec{r_0} + r\frac{d\vec{r_o}}{ds} \tag{2}$$

or

$$\vec{r}.\vec{\epsilon_0} = r\frac{dr}{ds} \tag{3}$$

that is:

$$\frac{dr}{ds} = cos\phi. \tag{4}$$

Therefore, denoting. $AM = r_1$ and $BM = r_2$ we will have:

$$\frac{dr_1}{ds} = cos\phi, \frac{dr_2}{ds} = cos(\pi - \phi), \tag{5}$$

that is

$$\frac{dr_1}{ds} = -\frac{dr_2}{ds}, or \ \frac{d(r_1 + r_2)}{ds} = 0, \tag{6}$$

and

$$r_1 + r_2 = const. \tag{7}$$

The formulation for the hyperbola can be.
Let $(c) = (c_a) \cup (c_b)$ be a smooth curve in E^2, where (c_a),(c_b) are convex and A,B are interior points in (c_a) and (c_b) respectivelly. The point $M \in (c)$ and we denote $\epsilon(M)$ the support line of the point M and $\vec{\epsilon_0}$ the unit tangent vector at the point M. We suppose that:

$$\angle(\vec{AM}, \vec{\epsilon_0}) = \angle(\vec{BM}, \vec{\epsilon_0}) \tag{8}$$

Then (c) must be an hyperbola.
The proof is exactly the same as in the ellipse.
The formulation for the parabola is the same with the ellipse except the compactness of (c) and supposing that $B = \infty$.
For the solution we point out that we determine the point O so that AO be perpendicular to the tangent to the point O of (c) and we choose OA as axis OX. Using Analytical Geometry we take an easy differential equation. We also can find a synthetic Geomtrical proof using the reduction ad absurdium method. That is, we consider the parabola (c_1) with vertex O and focus A and then we prove that $(c) = (c_1)$.