An interesting theorem in Topology.

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We will prove the following interesting theorem, similar to the Helly type theorems in Convexity.

Theorem

$F_1,\ F_2,\ F_3,..\quad ..F_{n+1}$ are $n+1$ point sets, no pair of them overlap, topological equivalent to B^n, so that $F_i \cap F_j \neq 0$ and $U = \cup F_i$ is topological equivalent to B^n as well, then it is: $\cap F_i \neq 0$ for, $i, j = \{1, 2, 3, ..\ \ n+1\}$.

Proof

Let the point $A_k \in intF_k$ and the point $B_{ij} \in F_i \cap F_j$. We join by path l_{ij} through B_{ij} the points $A_i \in intF_i,\ A_j \in intF_j$, so that $l_{ij} \in F_i \cup F_j$. We can suppose that the points A_i, $i = \{1, 2, 3..\ \ n+1\}$ are the vertices of a simplex $S_A^{(n)}$ in E^n. The vertices of the simplex are joint by the paths l_{ij} but we can consider $S_A^{(n)}$ as tological equivalent to a smplex $S_C^{(n)}$ with edges str.line segments. Therefore in the following we can use $S_A^{(n)}$, without any restriction to the proof, considering that the paths are str.line segments and we will denote $S^{(n)} = S_A^{(n)}$.
U is a topological equivalent to $B^{(n)}$ and $S^{(n)}$ is covered by U, that is $S^{(n)}$ is covered by the (non overlaping) point sets $F_1,\ F_2,..\quad F_{n+1}$. We now consider the following triangulation of $S^{(n)}$. We arbitrary take points P_i in $F_i \cap S^{(n)}$, that is all the points P in the $F_i \cap S^{(n)}$ are indexed by 'i'. According to the Sperner's lemma , at least a n-smplex of the triangulation will be $t_1 = P_1P_2P_3..\quad ..P_{n+1}$. A second triangulation (we take arbitrary points inside to t_1) gives one at least n-simplex $t_2 = P_1P_2..\quad ..P_{n+1}$, so that the Diameter of t_1 is no less than the Diameter of t_2 and $t_1 \supset t_2$. Therefore we

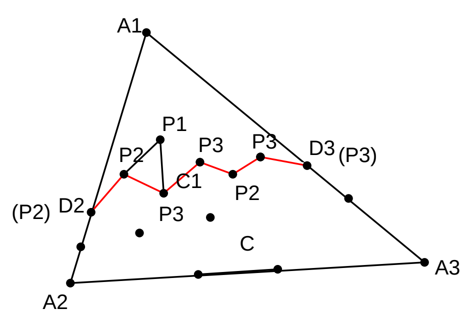
choose a sequence, according Bolzano-Weierstrass theorem $t_1, t_2, t_3, \ldots\ t_k, ..,$ so that $\cap_{i=1}^{\infty} F_i \neq 0$.
A remarkable proof of Sperner's lemma is the following.

Sperner,s lemma.

Let $S^{(n)}$ be a simplex in E^n. We consider the following triangulation: To the face A_j opposite to the vertex A_j we take arbitrary (relative to the position and the number) points denoted by P_i where $j \neq i = \{1, 2, ...j-1, j+1, ...n+1\}$. In the interior of $S^{(n)}$ we take arbitrary points $P_i,\ i = \{1, 2, 3, ...n+1\}$. Then there is at least one n-simplex $P_1 P_2 P_3 ..\ P_{n+1}$

Proof

We will use induction relative the dimension n. That is we assume that the proposition is correct for the dimension n-1. In the face a_i there are points denoted by $P_2, P_3, ..\ \ ..P_{n+1}$. We consider the max. complex C including points $P_i,\ i \neq 1$, and C_1 its subcomplex, so that its surface to be closest to A_1. To understand what I mean, see the figure for $E^{(2)}$. Suppose that C_1 has common points with the edges $A_1 A_i,\ \ i = \{2, 3, ..\ \ n+1\}$ the points $D_2, D_3, ..\ \ ..D_{n+1}$. We consider the $(n-1)$-simplex $d = D_2 D_3 ..\ \ ..D_{n+1}$ and we see that we can consider the points of C_1 as a tringulation of the simplex d.



Accepting that the lemma of Sperner in valid for E^{n-1} that is for the

simplex d we see that we can find in C_1 a simplex $d_1 = P_2P_3.. \quad ..P_{n+1}$. Therefore according our choice of C_1 the next point of the triangulation of the $S^{(n)}$ simplex is a point P_1. Hence we found a simplex $P_1P_2.. \; P_{n+1}$ of the triangulation.
As an interesting application of the theorem we prove below the theorem of Brouwer.

The theorem of the fixed point of Brouwer.
Let F be a point set tological equivalent to the $B^n = \{x/|x| \leq 1\}$ ball and q is a continuous transformation. Then $\exists x \in F$ so that: $q(x) = x$

The proof of the theorem of Brouwer.
The point set F is topological equivalent to $B^{(n)}$ ball, hence we can consider that F is the regular n-simplex $S^{(n)}$. Let O be the center of the $S^{(n)}$: $c = \{x/|x| = 1\}$ is a $S^{(n-1)}$ sphere (that is the surface of a $B^{(n)}$ ball) and we denote by $B_i = OA_i \cap c$, and by b_i the spherical $S^{(n-1)}$ simplex $b_i = B_1B_2....B_{i-1}B_{i+1}.. \; B_{n+1}$ opposite to the vertex B_i of the simplex $S_1^{(n)} = B_1B_2.. \; ..B_{n+1}$. The sector of the $B^{(n)}$ ball, $W = (O,1) = \{x/|x| \leq 1\}$, with base b_i is denoted by W_i. We see that $\cup b_i = \omega$ the "'area" ((n-1)volume) of the surface of the W sphere and $\cup W_i = volume \; W$.
Let now $x, q(x) \in S^{(n)}$. We suppose that $\vec{OD_i} = x\vec{q(x)}$ and the end point $D_i \in W_i$. It is remarkable to point aout here that each W_I characterizes a direction in the space. We also denote by U_i the point set of the end points $q(x)$. The transformation q is continuous therefore U_i is connect compact point set, topological equivalent to the $B^{(n)}$ sphere. It is not difficult to see that: $U_i \cap U_j \neq 0$. That is because there is a common vector $xq(x)$ to both the sectors W_i, W_j, therefore the end-point $q(x)$ is in $U_i \cap U_j$.
The point sets U_i realize the conditions of the theorem, therefore: $\cap_1^{n+1} U_i \neq 0$, that is for a common point to U_i there is a point D in all the sectors W_i (so $x\vec{q(x)}$ has every direction in the space), hence there is point $x \in S^{(n)}$, so that $x = q(x)$.

Remark
Combining our theorem and the theorem of Helly we can obtain the following remarkable theorem.

Theorem

$F_1,\ F_2,\ F_3, ..\ \ ..F_k$ are k, $k \geq n+1$, point sets, no pair overlap, topological equivalent to B^n, so that $F_i \cap F_j \neq 0$ and the union of every $n+1$ of them is topological equivalent to B^n as well, then it is: $\cap_1^k conv(F_i) \neq 0$.

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