The triangle of minimum perimeter circumscribed to a smooth closed compact convex curve in the plane.

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Introduction

It is well known that there is a triangle of minimum area circumscribed to a smooth closed compact convex curve (c) in E^2, see [1],[2]. In this note we solve the analogous problem for the perimeter. Theorems of the Analysis assure us that there is at least one triangle T of minimum perimeter circumscribed to (c), see [3]. We proved the following theorem.

Theorem

For every triangle T circumscribed to (c) the excircles of T are tangent to (c). **Proof**

We proceed by contraposition. We suppose that the triangle T=ABC, of minimum perimeter, is circumscribed to (c) and (I_a, r_a) the excircle to BC. We denote by D the tangent point of (I_a, r_a) to BC by W the tangent point of BC to (c) and by M the tangent point on AC. From the elementary Geometry we know that:

$$AM = \frac{1}{2}(AB + BC + CA) \tag{1}$$

We assume that $W \neq D$ see fig(1).

We consider now a movement of a circle (I,r) from the position of (I_a, r_a) towards the (c), such a way that (I,r) remains tangent to the sides of the angle BAC and homothetic to (I_a, r_a) with center of Homothesy A and ratio $\frac{r}{r_a} = k(variable) < 1$.

The final position of the circle (I,r) will be the circle (I_0, r_0) which is tangent to (c) at the point W_0 and tangent to the sides of the angle BAC

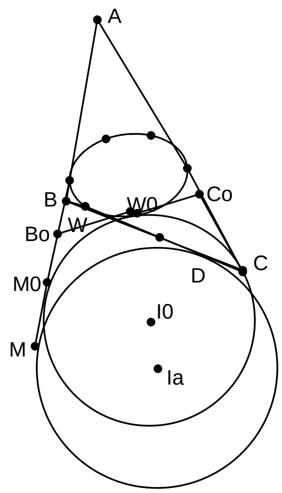

Figure 1:

(M_0 $on AB$), see fig (1). The triangle ABC will be the triangle AB_0C_0 and its perimeter is:

$$AM_0 = \frac{1}{2}(AB_0 + B_0C_0 + C_0A). \tag{2}$$

Obviously $AM > AM_0$ and from (1) and (2) follows the contradiction.
A second **Proof**
We can also obtain a second and very interesting proof using Differential Geometry. We denote:

$$|\vec{AB}| = c, \quad \frac{\vec{AB}}{c} = \vec{c}_0, \quad |\vec{AC}| = b, \quad \frac{\vec{AC}}{c} = \vec{b}_0$$

$$|\vec{BC}| = a \quad \frac{\vec{BC}}{a} = \vec{a}_0, \quad |\vec{WB}| = x_1, \quad |\vec{WC}| = x_2.$$

see fig 2.

Let O be the origin and Q a point of (c). The posotion of a point W depends on the arc length s from Q to W. We parametrize on the (c) counterclockwise.

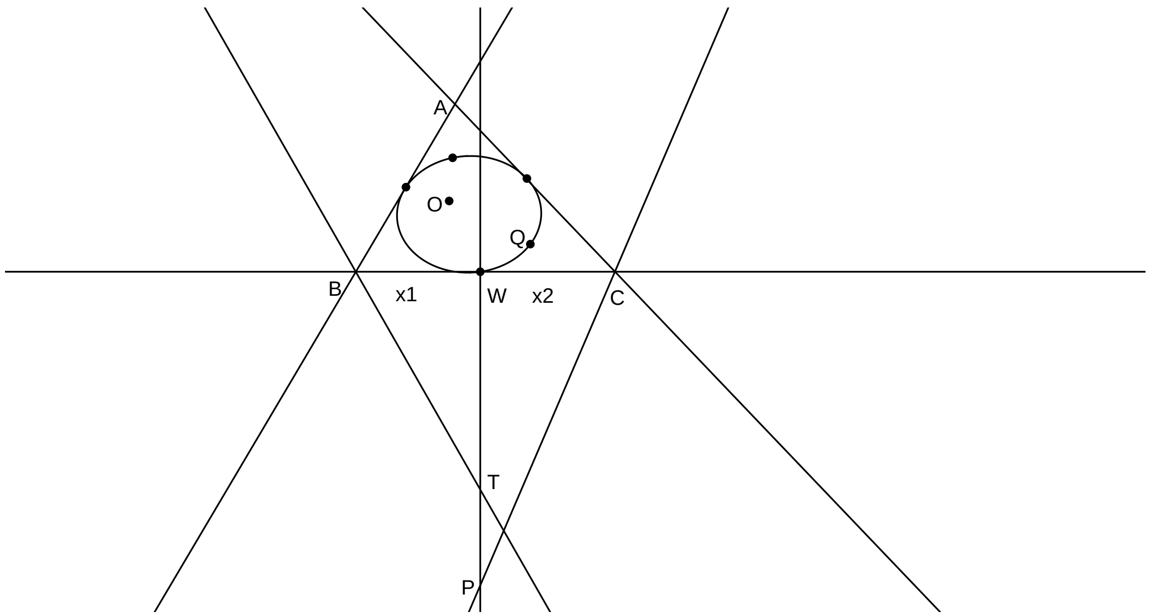

Figure 2:

Let $\vec{r}(s) = \vec{OW}$. We have:

$$\vec{OW} + \vec{WB} = \vec{OA} + c \cdot \vec{c_0}, \quad or$$

$$\vec{r}(s) - x_1 \frac{d\vec{r}}{ds} = \vec{OA} + c \cdot \vec{c_0}$$

$$\vec{r}(s) - x_1 \cdot \vec{\epsilon_0} = \vec{OA} + c \cdot \vec{c_0}. \tag{3}$$

We denote $\vec{\epsilon_0}, \quad \vec{\eta_0}$ the unit tangent and the principal normal, from Frenet formulas and from (3) follows:

$$\dot{\vec{r}}(s) - \dot{x_1}\vec{\epsilon_0} - x_1\dot{\vec{\epsilon_0}} = \dot{c} \cdot \vec{c_0}, \quad or$$

$$\vec{\epsilon_0} - \dot{x_1}\vec{\epsilon_0} - x_1 \cdot k \cdot \vec{\eta_0} = \dot{c} \cdot \vec{c_0}. \tag{4}$$

Where k the curvature of (c) at the point W. We multiplay (4) succesively by $\vec{\epsilon_0}, \quad \vec{\eta_0}$. We take:

$$1 - \dot{x}_1 = \dot{c}(-\ cosB) \tag{5}$$

$$-x_1 \cdot k = \dot{c}\ sinB. \tag{6}$$

Therefore:

$$\frac{d(c+x_1)}{ds} = \dot{c} + \dot{x}_1 = \frac{x_1 \cdot k}{\sin B} + 1 - \frac{x_1 \cdot k}{\sin B}\cos B$$

or

$$\frac{d(c+x_1)}{ds} = 1 - x_1 \cdot k\ cotg\frac{B}{2}. \tag{7}$$

Similarly, we find:

$$\frac{d(b+x_2)}{ds} = -(1 - x_2 \cdot k\ cotg\frac{C}{2}). \tag{8}$$

From (7),(8) we take:

$$\frac{d(a+b+c)}{ds} = -x_1 \cdot k\ cotg\frac{B}{2} + x_2 \cdot k\ cotg\frac{C}{2}.$$

So to minimize a+b+c:

$$\frac{d(a+b+c)}{ds} = 0$$

or

$$x_1\ cotg\frac{B}{2} = x_2\ cotg\frac{C}{2}, \tag{9}$$

This last equation (9) says that the perpendicular to BC at the point W intersects the bissectrices of the exterior angles of B and C at the points T and P so that WT=WP. Therefore the excircle of BC is tangent to (c) at the point W.

Remarks

1. The above theorem can be easily generalized to a n-gon circumscribed to (c), e.g. for a 6-gon $P_6 = A_1A_2A_3A_4A_5A_6$. We project the sides and externaly to P_6 we take successively the triangles $T_1 = A_1B_1A_2,\quad T_2 = A_2B_2A_3,\quad .....T_6 = A_6B_6A_1(A_1A_6 \cap A_2A_3 = B_1,\ etc)$. For the 6-gon of a minimum perimeter circumscribed to (c) the incircles in T_i must be tangent to (c).

2. It would be quite interesting an analogous problem to E^3, that is: Find the tetrahedron of a minimum surface circumscribed to a convex body.

3 The proved theorem in E^2 , with the above two proofs, has been submitted by the author to Crux as a problem dedicated to the memory of my friend Murray Klamkin. Here it is published by the permission of Crux.

References

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Math.Magazine, vol44, No 2, pp 57-69.
2. M. M. Day, Polygons circumscribed about closed convex curves, Trans. Am. Soc. Vol 62, pp 315-319, 1957.
3. Crux, September 2005, KLAMKIN 13, submitted by George Tsintsifas and solved by Yufel Zhao in Crux September 2006.